Loughborough University

INTRODUCTION TO DYNAMICAL SYSTEMS
(21MAC148)

Semester 1 2021 (1a) Exam paper

Answer ALL questions. Show all your working.

This is a (1a) online examination, meaning you have 23 hours in which to complete and submit this paper. How you manage your time within the 23-hour window is up to you, but we expect you should only need to spend approximately 3 hours working on it. If you have extra time or rest breaks as part of a Reasonable Adjustment, you will need to add this to the amount of time you are expected to spend on the paper.

It is your responsibility to submit your work by the deadline for this examination. You must make sure you leave yourself enough time to do so.

It is also your responsibility to check that you have submitted the correct file.

Exam Help

If you are experiencing difficulties in accessing or uploading files during the exam period you should contact the exam helpdesk. For urgent queries please call **01509 222900**. For other queries email examhelp@lboro.ac.uk

You may handwrite, LaTeX and/or word process your answers, as you see fit.

You may use any calculator (not just those on the University's approved list).

1. (a) Consider a map $f : I \to \mathbb{R},\ I \subseteq \mathbb{R}$. Explain whether or not this map determines a dynamical system according to the formula $x_{n+1} = f(x_n),\ \ x \in I$ for the following cases:

   (i) $f(x) = (|1-x| - |1+x|)/2,\ \ I = [-1, 1]$; [3]

   (ii) $f(x) = (\sin 3x + 3\sin x)/2,\ \ I = [-\pi/2, \pi/2]$. [6]

   (b) Find all the fixed points of the dynamical system

$$x_{n+1} = 0.5x_n(\cos 2x_n - 3\cos x_n + 4),\ \ x \in [-5\pi/12, \pi/4].$$

   Investigate stability of the fixed point(s) for which the multiplier is different from 1. [6]

   (c) Draw the cobweb plot and use this plot to establish whether the fixed point $x = 0$ is an attractor, or a repeller, or neither of them for the dynamical system
   $x_{n+1} = -x_n + x_n^2,\ x \in \mathbb{R}$. [5]

   (d) Consider the dynamical system

$$x_{n+1} = \frac{a}{1+x_n^2} - \frac{x_n}{2},\ \ x \in \mathbb{R},$$

   where a is a constant. Determine values of a such that $\{\overline{0, a}\}$ is a period-2 orbit of this dynamical system. Determine whether these periodic orbits are attractors or repellers.

   [5]

2. The base-5 shift map $f : [0,1) \to [0,1)$ is defined by the formula

$$f(x) = \mathrm{Frac}(5x),$$

where $\mathrm{Frac}(u)$ denotes the fractional part of u.

(a) Define the sequence space Σ_5 and the map $\sigma : \Sigma_5 \to \Sigma_5$ such that the action of σ on sequences in Σ_5 coincides with the action of f on sequences of digits in the base-5 representation of numbers in the interval $[0,1)$. [3]

(b) (i) Determine all the fixed points for σ, and hence using the symbolic dynamics determine all the fixed points for f, expressing your answers in symbolic form for σ and as fractions for f. [3]

(ii) Find the minimal and the maximal values $x \in (0,1)$ such that x is a period-4 point of f. [5]

(iii) How many periodic points of the minimal period 8 do the maps σ and f have? Justify your answers. [5]

(c) Let $\mathbf{s} = (s_1 s_2 \ldots)$ and $\mathbf{t} = (t_1 t_2 \ldots)$ be two points of Σ_4. The distance in Σ_5 between $\mathbf{s}$ and $\mathbf{t}$ can be defined as $d[\mathbf{s},\mathbf{t}] = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{5^i}$.

(i) Prove that if $s_i = t_i$ for $1 \le i \le n$, then $d[\mathbf{s},\mathbf{t}] \le 1/5^n$. [3]

(ii) Prove that if $d[\mathbf{s},\mathbf{t}] < 1/5^n$, then $s_i = t_i$ for $1 \le i \le n$. [2]

(iii) A map $\psi : \Sigma_5 \to \Sigma_5$ is said to be uniformly continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\mathbf{s} \in \Sigma_5, \mathbf{t} \in \Sigma_5$ with $d[\mathbf{s},\mathbf{t}] < \delta$ we have $d[\psi(\mathbf{s}),\psi(\mathbf{t})] < \varepsilon$. Prove that the map σ^3 is a uniformly continuous map. [4]

3. (a) Consider the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\sqrt{4+x^2+y} + x^2 - 1 + x \\ 5\ln(x^2-3) + y \end{pmatrix}.$$

(i) Find all the fixed points of this map. [6]

(ii) Study stability of the fixed points of this map. [7]

(b) Find the area in $\mathbb{R}^2$ of the image of the set $x^2 + 4x - 8y + 4y^2 \leq 0$ under the transformation by shift along solutions of the system

$$\dot{x} = x\cosh^3 t - y\cosh^2 t\,,\ \dot{y} = x\sinh^2 t + y\sinh^3 t$$

from time $t = 0$ until time $t = \ln 2$. [9]

(c) Explain whether or not there exists a linear homogeneous time-periodic ODE for which the matrix

$$\begin{pmatrix} 16 & 4 \\ -4 & -1 \end{pmatrix}$$

is a matrix of the monodromy map? [3]

4. Draw bifurcation diagrams for the following systems depending on the parameter α. Indicate a standard name for each of the observed bifurcations.

   (a) $\dot{x} = -2x^2 - \frac{x^4}{2+x^2} - \alpha - 2.$ [5]

   (b) $\dot{x} = (x - 2\alpha)(x^2 + x - 1 + \alpha).$ [6]

   (c) $\dot{x} = x\,(1 + \alpha - \sinh x).$ [4]

   (d) $\dot{r} = (\alpha + \alpha^3 - 2)r - r^3,\ \dot{\varphi} = 1 + r^2,$
   where r, φ are polar coordinates. [4]

   (e) $\dot{x}_1 = \alpha - 1 + \cosh x_1,\ \dot{x}_2 = \alpha + \alpha^5 - x_2.$ [6]