Loughborough University

PROBABILITY THEORY
(21MAB170)

January 2022 1(a) Exam

This is a 1(a) remote assessment examination meaning you have **23 hours** in which to complete and submit this paper. How you manage your time within the 23-hour window is up to you but we expect you should only need to spend approximately **2 hours** working on it. If you have extra time or rest breaks as part of a Reasonable Adjustment, you will need to add this to the amount of time you are expected to spend on the paper

It is your responsibility to submit your work by the deadline for this examination. You must make sure you leave yourself enough time to do so

It is also your responsibility to check you have submitted the correct file.

Exam help

If you are experiencing difficulties in accessing or uploading files during the exam period you should contact the exam helpdesk. For urgent queries please call **01509 222 900**.

For other queries e-mail `examhelp@lboro.ac.uk`

You may handwrite your answers.

You may use any calculator (not just those on the University's approved list).

Answer **ALL FOUR** questions.

---

1. A pair of jointly continuous random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 2x\mathrm{e}^{-x-2y}, & x,y > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find marginal density functions for X and Y. [6]

(b) Show that $\mathbb{E}(XY) = 1$ and hence calculate $\mathrm{Cov}(X,Y)$. Interpret your answer. [9]

(c) Show that a probability density function for $X+Y$ is

$$f(x) = 2(x-1)\mathrm{e}^{-x} + 2\mathrm{e}^{-2x}, \qquad x > 0,$$ [7]

and hence calculate $\mathbb{P}(X+Y > 1)$. [3]

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

   (a) Let $A, E_1, \ldots, E_n \in \mathcal{F}$ with $\mathbb{P}(E_i) > 0$ for each $i = 1, \ldots, n$.

      (i) State what it means for $\{E_1, \ldots, E_n\}$ to be a partition of Ω. [4]

      (ii) If $\{E_1, \ldots, E_n\}$ is a partition, prove that

$$\mathbb{P}(A) = \mathbb{P}(A \mid E_1)\mathbb{P}(E_1) + \cdots + \mathbb{P}(A \mid E_n)\mathbb{P}(E_n).$$ [6]

   (b) Two urns both contain 5 red balls and 4 blue balls, and a third urn contains 7 red balls and 2 blue balls. One of the three urns is chosen uniformly at random, and a ball is chosen uniformly at random from that urn. Using the result above, or otherwise, find the probability that the chosen ball is red. [8]

   (c) If $B \in \mathcal{F}$ with $0 < \mathbb{P}(B) < 1$, prove that if $\mathbb{P}(A \mid B) = \mathbb{P}(A \mid B^c)$ then A and B are independent. (You may use without proof the fact that $\mathbb{P}(B) + \mathbb{P}(B^c) = 1$.) [7]

3. (a) Define what is meant by the moment-generating function $M(t)$ for a random variable, and explain why $M(0) = 1$. [4]

   (b) A random variable X has moment-generating function

$$M_X(t) = C\frac{3-t}{(4-t)(2-t)}, \qquad t < 2$$

   where C is a constant.

      (i) Find the value of the constant C. [2]

      (ii) Find the mean and the variance of X (Hint: use partial fractions...). [11]

   (c) If U and V are independent random variables with moment-generating functions, use moment-generating functions to prove that

$$\mathrm{Var}(U + V) = \mathrm{Var}(U) + \mathrm{Var}(V).$$ [8]

4. (a) (In this part you may use the summation formula

$$\sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2}, \qquad |a| < 1.)$$

A pair of jointly distributed discrete random variables have joint probability mass function

$$p_{X,Y}(n,m) = \begin{cases} \frac{5}{27}\left(\frac{2}{3}\right)^{n+m}, & n \geq m \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the marginal mass function for Y and find $\mathbb{E}(Y)$. [7]

(ii) Show that the conditional probability mass function of X given event $\{Y = m\}$, $m \geq 0$, is

$$p_{X|Y}(n \mid m) = \frac{1}{3}\left(\frac{2}{3}\right)^{n-m}, \qquad n \geq m.$$ [4]

(iii) Find the conditional expectation $\mathbb{E}(X \mid Y)$. [5]

(iv) Use your answer to part (iii) to find $\mathbb{E}(X)$. [3]

(b) Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain on two states $\{0, 1\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

(i) Prove by induction that

$$\mathbf{P}^n = \begin{pmatrix} \frac{1}{2}(1+2^{-n}) & \frac{1}{2}(1-2^{-n}) \\ \frac{1}{2}(1-2^{-n}) & \frac{1}{2}(1+2^{-n}) \end{pmatrix}$$ [3]

(ii) Hence calculate $\mathbb{P}(X_2 = 1 \mid X_0 = 0)$ and $\mathbb{P}(X_{10} = 1 \mid X_0 = 0)$. [3]