MODERN OPTICS
(21PHC108)

Semester 2 2022 — In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

Answer **QUESTION 1 and TWO others.**

A formula sheet is provided on pages 4-7.

1. You may answer as many parts of Question 1 as you wish. All work you do will be assessed and the marks totalled but note that the maximum total credit for this question will be 20 marks.

   (a) For aluminium at $\lambda = 500$ nm the real and imaginary parts of the index of refraction $n = 1.5$ and $\kappa = 3.2$. Find the reflectance at normal incidence. [5]

   (b) A He-Ne laser has been designed to operate between two Brewster windows, in addition to the optical resonator. Explain the resulting polarisation of the laser light. [5]

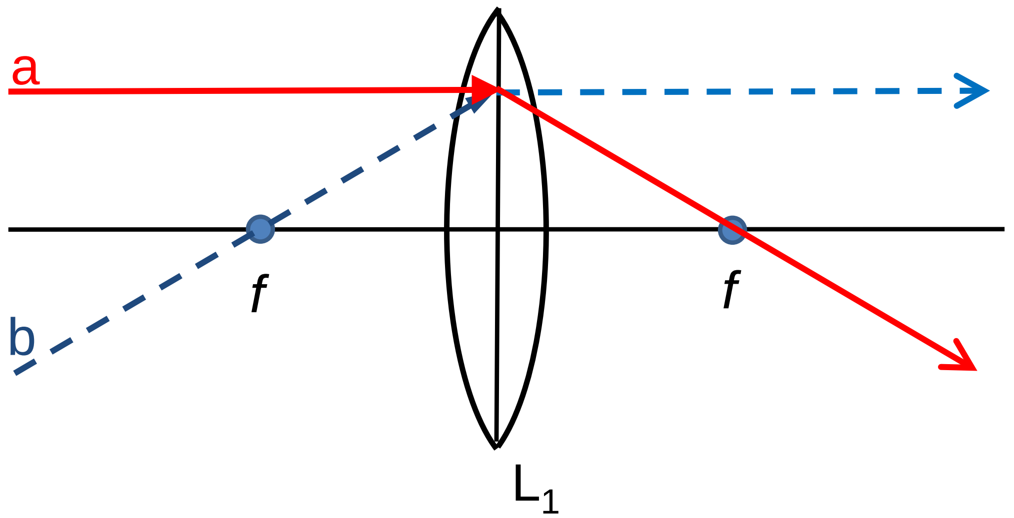
   (c) Discuss the interaction length for second harmonic interaction for cases with and without velocity matching. [5]

   (d) Explain the existence of radiation pressure in terms of the quantum description of light. Derive the relation between light pressure and light intensity. [5]

   (e) Discuss the working principle of a beam splitter cube. [5]

   (f) A Fresnel zone plate was made from a polarising synthetic sheet (polaroid film), in such way that the light is polarised vertically in all even zones and horizontally in all odd zones. Explain the principle of operation for such zone plate. [5]

2. (a) Use the rays illustrated in the diagram to help determine the ABCD transmission matrix for the biconvex lens with the focal length, f, in the paraxial regime. [8]



   (b) Derive the Cauchy's equation $n(\lambda_0) = A + D/\lambda_0^2$, where A and D are constants, from the general formula for dispersion in isotropic dielectrics. [6]

   (c) A filter is used to obtain approximately monochromatic light from a white source. If the pass band of the filter is 10 nm, what is the coherence length and coherence time of the filtered light? The mean wavelength is 600 nm. [6]

3. (a) What is the minimal resolving power a diffraction grating should have in order to resolve Raman lines of $CsCl_4$ excited with 435.83 nm wavelength, if the molecule's vibration frequencies correspond to wavenumbers of 217 cm^{-1} and 315 cm^{-1}? [9]

   (b) Obtain the relationship between the phase velocity, v, and the group velocity, v_g, if the corresponding dispersion is $v = a/\lambda$, where a is a constant. [5]

   (c) Consider a 1D photonic crystal made from alternating dielectric layers of thicknesses $d_1 = 100$ nm and $d_2 = 175$ nm and with refraction indices $n_1 = 1.5$ and $n_2 = 2$, respectively. Calculate the central frequencies of the first and second photonic bandgaps. [6]

4. (a) A laser that emits a diffraction-limited beam ($\lambda_0 = 632.84$ nm), produces a light spot on the surface of the Mooon a distance of $376,000$ km away. How big is the circular aperture of the laser, if the light spot has a radius of 58 km? Neglect any effects of the Earth's atmosphere. [8]

   (b) Consider a ruby crystal with two energy levels separated by an energy difference corresponding to a free-space wavelength $\lambda_0 = 694.3$ nm, with a Lorentzian lineshape of width $\Delta\nu = 330$ GHz. The spontaneous lifetime is $t_{sp} = 3$ ms and the refractive index of ruby is $n = 1.76$. What value should the population difference $N_2 - N_1$ assume to achieve a gain coefficient $\gamma(\nu_0) = 0.5$ cm^{-1} at the central frequency? [8]

   (c) How long should the crystal be to provide an overall gain of 10 at the central frequency when $\gamma(\nu_0) = 0.5$ cm^{-1}? [4]

Formula Sheet

$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{r}$	Spherical surface
$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$	Gaussian lens equation
$\frac{1}{s} + \frac{1}{s'} = \frac{2}{r}$	Spherical mirror
$0 < d < 4f$	Stability criterion for an optical resonator from two concave mirrors
$T = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$	Ray tracing for a thin lens
$\frac{1}{f_N} = \frac{1}{f_1} + \frac{1}{f_2} + .. + \frac{1}{f_N}$	System of thin lenses
$c = 1/\sqrt{\epsilon_0 \mu_0} \approx 3 \times 10^8$ m/s	velocity of light in vacuum
$h = 6.62 \times 10^{-34}$ J·s	Planck constant
$\epsilon_0 = 8.85 \times 10^{-12}$ F/m=A·s/(V·m)	permittivity of free space
$\mu_0 = 4\pi \times 10^{-7}$ H/m=V·s/(A·m)	permeability of free space
A=kg·m^2·s^{-3}·V^{-1} =W·V^{-1}	conversion between Ampere and Volt
$v = \nu\lambda,\ k = 2\pi/\lambda,\ N = 1/\lambda$	conversion of speed, wavelength, frequency, angular wavenumber and wavenumber
$v = \omega/k = 1/\sqrt{\epsilon\mu} = c/n$	phase velocity of EM wave
$v_{\mathrm{g}} = d\omega/dk$	group velocity
$v_{\mathrm{g}} = v - \lambda \frac{dv}{d\lambda}$	group velocity
$\frac{1}{v_{\mathrm{g}}} = \frac{1}{v} - \frac{\lambda_0}{c}\frac{dn}{d\lambda_0},\ \lambda_0 = \lambda \cdot n$	group velocity
$\vec{\nabla} \cdot \vec{D} = \rho,\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Maxwell's equations
$\vec{\nabla} \cdot \vec{B} = 0,\ \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$	Maxwell's equations
$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$	relation between $\vec{D}$, $\vec{E}$ and $\vec{P}$
$\vec{P} = \chi \epsilon_0 \vec{E}$	relation between $\vec{P}$ and $\vec{E}$
$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu \vec{H}$	relation between $\vec{B}$, $\vec{H}$ and $\vec{M}$
$\vec{M} = \chi_{\mathrm{m}} \vec{H}$	relation between $\vec{M}$ and $\vec{H}$

$\omega\mu\vec{H} = \vec{k}\times\vec{E}$	$\vec{H}$ and $\vec{E}$ of an EM wave
$-\omega\epsilon\vec{E} = \vec{k}\times\vec{H}$	$\vec{H}$ and $\vec{E}$ of an EM wave
$\vec{S} = \vec{E}\times\vec{H}$	Poynting vector
$I = \langle S\rangle = v\epsilon E_0^2/2$	Irradiance
$P = I/v$	Light pressure
$\vec{p} = \hbar\vec{k}$	Photon momentum
$n_\mathrm{i}\sin\theta_\mathrm{i} = n_\mathrm{t}\sin\theta_\mathrm{t}$	Snell's law
$\vec{E}(\vec{r},t) = \vec{E}_0\cos(\vec{k}\cdot\vec{r}-\omega t+\phi)$	Description of E-field
$\alpha_\mathrm{max} = \sqrt{n_1^2-n_2^2}$	Fibre optics acceptance angle
$I(\theta) = I(0)\cos^2\theta$	Intensity after a polariser

Amplitude reflection and transmission coefficients

$$r_\perp = \frac{n_\mathrm{i}\cos\theta_\mathrm{i}-n_\mathrm{t}\cos\theta_\mathrm{t}}{n_\mathrm{i}\cos\theta_\mathrm{i}+n_\mathrm{t}\cos\theta_\mathrm{t}} = -\frac{\sin(\theta_\mathrm{i}-\theta_\mathrm{t})}{\sin(\theta_\mathrm{i}+\theta_\mathrm{t})}$$

$$t_\perp = \frac{2n_\mathrm{i}\cos\theta_\mathrm{i}}{n_\mathrm{i}\cos\theta_\mathrm{i}+n_\mathrm{t}\cos\theta_\mathrm{t}} = \frac{2\sin\theta_\mathrm{t}\cos\theta_\mathrm{i}}{\sin(\theta_\mathrm{i}+\theta_\mathrm{t})}$$

$$r_\parallel = \frac{n_\mathrm{i}\cos\theta_\mathrm{t}-n_\mathrm{t}\cos\theta_\mathrm{i}}{n_\mathrm{t}\cos\theta_\mathrm{i}+n_\mathrm{i}\cos\theta_\mathrm{t}} = -\frac{\tan(\theta_\mathrm{i}-\theta_\mathrm{t})}{\tan(\theta_\mathrm{i}+\theta_\mathrm{t})}$$

$$t_\parallel = \frac{2n_\mathrm{i}\cos\theta_\mathrm{i}}{n_\mathrm{i}\cos\theta_\mathrm{t}+n_\mathrm{t}\cos\theta_\mathrm{i}} = \frac{2\sin\theta_\mathrm{t}\cos\theta_\mathrm{i}}{\sin(\theta_\mathrm{i}+\theta_\mathrm{t})\cos(\theta_\mathrm{i}-\theta_\mathrm{t})}$$

$\tan\theta_\mathrm{B} = n_\mathrm{t}/n_\mathrm{i}$	Brewster's angle
$R = r^2,\ T = \left(\frac{n_\mathrm{t}\cos\theta_\mathrm{t}}{n_\mathrm{i}\cos\theta_\mathrm{i}}\right)t^2$	Reflectance and transmittance
$\mathcal{N}^2 = 1-\omega_\mathrm{p}^2/(\omega^2+\mathrm{i}\omega/\tau),\ \omega_\mathrm{p} = \sqrt{Nq_\mathrm{e}^2/(m_\mathrm{e}\epsilon_0)}$	Dispersion for a metal
$\mathcal{N}^2 = 1+(Nq_\mathrm{e}^2/m_\mathrm{e}\epsilon_0)\sum_j f_j/(\omega_{0j}^2-\omega^2-\mathrm{i}\gamma_j\omega)$	Dispersion for a dielectric
$\mathcal{N} = n+\mathrm{i}\kappa$	Complex refractive index
$R = \left(\frac{\mathcal{N}-1}{\mathcal{N}+1}\right)\left(\frac{\mathcal{N}^*-1}{\mathcal{N}^*+1}\right),\ \mathcal{N} = n+\mathrm{i}\kappa$	Normal reflectance from a metal
$n_\mathrm{e} = \sqrt{1+\chi_{33}}$	Extraordinary index of refraction
$n_\mathrm{o} = \sqrt{1+\chi_{11}}$	Ordinary index of refraction
$\theta = \pi l(n_\mathrm{R}-n_\mathrm{L})/\lambda = \pi l\chi_{12}/(n_\mathrm{o}\lambda)$	Rotatory power of an optically active medium
$\Lambda = \sum_i l_i n_i$	Optical path length
$\Delta\phi = k_0\Delta\Lambda = \frac{2\pi}{\lambda_0}\Delta\Lambda$	Relative phase difference

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$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\Delta\phi$	Interference of two waves
$\Delta\Lambda = 2nd\cos\theta$	Optical path difference in Michelson interferometer
$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\mathrm{Re}(\gamma_{12}(\tau))$	Interference (partial coherence)
$l_c = \frac{c}{\Delta\nu} = c\Delta\tau$	Coherence length
$\Delta\Lambda = 2n_{\mathrm{t}}d\cos\theta_{\mathrm{t}}$	Optical path difference in thin films
$I = I_{\mathrm{max}}/(1 + F\sin^2(k_0\Delta\Lambda + \delta_{\mathrm{r}})/2)$	Fabry-Perot interferometer
$F = 4r^2/(1-r^2)^2$	Coefficient of finesse
$P = \epsilon_0(\chi E + \chi^{(2)}E^2 + \chi^{(3)}E^3 + ...)$	Non-linear polarisation expansion for isotropic media
$I^{(2\omega)} \propto [\sin(2\pi(n_\omega - n_{2\omega})l/\lambda_0)]^2 \,/\, [2\pi(n_\omega - n_{2\omega})l/\lambda_0]^2$	Intensity of the second harmonic generated in a crystal slab
$M = \frac{1}{2n_1}\begin{pmatrix} n_1+n_0 & n_1-n_0 \\ n_1-n_0 & n_1+n_0 \end{pmatrix}$	Transfer matrix between layers with n_0 and n_1
$M = \begin{pmatrix} \exp(-\mathrm{i}\phi) & 0 \\ 0 & \exp(\mathrm{i}\phi) \end{pmatrix}$, $\phi = n_1 k_0 l$	Transfer matrix for propagation through n_1
$\cos(\mathcal{K}\Lambda) = \frac{(n_1+n_2)^2}{4n_1n_2}\cos(\pi\frac{\omega}{\omega_B}) - \frac{(n_1-n_2)^2}{4n_1n_2}\cos(\zeta\pi\frac{\omega}{\omega_B})$	Dispersion relation for 1-dimensional photonic crystal
$\omega_B = \frac{\pi c}{n_1d_1 + n_2d_2}$	Bragg frequency
$\zeta = \frac{n_1d_1 - n_2d_2}{n_1d_1 + n_2d_2}$	
$U_{\mathrm{p}} = \frac{\mathrm{i}kU_0\mathrm{e}^{-\mathrm{i}\omega t}}{4\pi}\int\int_\Sigma \frac{\mathrm{e}^{\mathrm{i}k(r+r')}}{rr'}\left[\cos(\vec{n},\vec{r}) - \cos(\vec{n},\vec{r'})\right]\mathrm{d}A$	Fresnel-Kirchhoff integral formula
$\frac{1}{2}\left(\frac{1}{d'} + \frac{1}{d}\right)\delta^2 \ll \lambda$	Condition for Fraunhofer diffraction
$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2$, $\beta = \frac{1}{2}kb\sin\theta$	Single slit, Fraunhofer diffraction
$\theta = \frac{1.22\lambda}{D}$	Angular radius of the Airy disk
$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2 \cdot \cos^2\gamma$, $\gamma = \frac{1}{2}kh\sin\theta$	Double slit, Fraunhofer diffraction
$I(\theta) = I(0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\gamma}{N\sin\gamma}\right)^2$	Multiple slits, Fraunhofer diffraction
$h\sin\theta_m = m\lambda$	Diffraction grating maxima
$\frac{\lambda}{\Delta\lambda} = Nm$	Resolving power of a grating
$U(k_x,k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\mathrm{e}^{\mathrm{i}(k_xx+k_yy)}\mathrm{d}x\mathrm{d}y$	Fourier optics

$U_\mathsf{p} = \dfrac{U_0}{1+\mathrm{i}}\left[C(v)+\mathrm{i}S(v)\right]\Big\vert_{v_1}^{v_2},\ v_{1,2} = y_{1,2}\sqrt{k/(\pi L)}$	Single slit, Fresnel diffraction
$C(s) = \displaystyle\int_0^s \cos\frac{\pi v^2}{2}\mathrm{d}v$	Fresnel integrals
$S(s) = \displaystyle\int_0^s \sin\frac{\pi v^2}{2}\mathrm{d}v$	
$R_m = \sqrt{m\lambda L},\ 1/L = 1/h + 1/h'$	Fresnel zones
$\langle n_\nu\rangle = \dfrac{1}{\exp\left(h\nu/k_B T\right)-1}$	Bose-Einstein distribution for photons
$u_\nu = \dfrac{8\pi h\nu^3}{c^3}\dfrac{1}{\exp\left(h\nu/k_B T\right)-1}$	Planck formula
$I_\nu = \dfrac{2\pi h\nu^3}{c^2}\dfrac{1}{\exp\left(h\nu/k_B T\right)-1}$	Planck formula
$A_{21}/B_{21} = 8\pi h\nu^3/c^3$	Einstein coefficients
$A_{21} = 1/t_\mathsf{sp}$	
$\sigma(\nu) = (c/n)^2 g(\nu)/(8\pi\nu_0^2 t_\mathsf{sp})$	Effective cross-section for absorption
$g(\nu) = \dfrac{\Delta\nu/2\pi}{(\nu-\nu_0)^2+(\Delta\nu/2)^2}$	Lorentzian lineshape
$\gamma(\nu) = (N_2 - N_1)\sigma(\nu)$	Gain coefficient
$\gamma(\nu) = \dfrac{1}{x}\ln\left(I(x)/I_0\right)$	Gain coefficient definition
$G(\nu) = \exp(\gamma(\nu)d)$	Gain
$\nu_m = vm/(2d)$	Longitudinal laser modes
$\alpha_\mathsf{r} = \alpha_\mathsf{s} + \dfrac{1}{2d}\ln\dfrac{1}{R_1R_2}$	Total laser loss
$(N_2 - N_1)_\mathsf{t} = \alpha_\mathsf{r}/\sigma(\nu)$	Threshold population difference

Trigonometric relations

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$	$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$	$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$
$\cos^2(\alpha/2) = (1+\cos\alpha)/2,$	$\sin^2(\alpha/2) = (1-\cos\alpha)/2$
$\sin\alpha\cos\beta =$	$(1/2)\cdot(\sin(\alpha+\beta)+\sin(\alpha-\beta))$
$\sin\alpha = (\mathrm{e}^{\mathrm{i}\alpha}-\mathrm{e}^{-\mathrm{i}\alpha})/2\mathrm{i},\ \cos\alpha = (\mathrm{e}^{\mathrm{i}\alpha}+\mathrm{e}^{-\mathrm{i}\alpha})/2$	$\mathrm{e}^{\pm\mathrm{i}\alpha} = \cos\alpha \pm \mathrm{i}\sin\alpha$