PROBABILITY THEORY
(22MAB170)

Semester 1 2022/2023 — In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are **not** allowed).

Answer **THREE** questions.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
   (a) State what it means for $\mathbb{P}$ to be a probability measure. [4]
   (b) Let E_1 and E_2 be events in $\mathcal{F}$. State, but do not prove, a formula for $\mathbb{P}(E_1 \cup E_2)$ and use it to show that
   $$\mathbb{P}(E_1 \cup E_2) \leq \mathbb{P}(E_1) + \mathbb{P}(E_2).$$ [3]
   (c) Let E_1 and E_2 be events in $\mathcal{F}$. Prove that
   $$\mathbb{P}(E_1 \cup E_2) \geq \mathbb{P}(E_1) \quad \text{and} \quad \mathbb{P}(E_1 \cup E_2) \geq \mathbb{P}(E_2).$$
   (Hint: Begin by proving that if $A \subseteq B$ are events in $\mathcal{F}$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.) [5]
   (d) (i) Suppose that $\mathbb{P}(E_1) = 0.3$ and $\mathbb{P}(E_2) = 0.5$, find the maximum and minimum possible values of $\mathbb{P}(E_1 \cup E_2)$. [2]
   (ii) Suppose that $\mathbb{P}(E_1) = 0.7$ and $\mathbb{P}(E_2) = 0.5$, find the maximum and minimum possible values of $\mathbb{P}(E_1 \cup E_2)$. [2]
   (e) If $\mathbb{P}(E_2) > 0$, prove that $0 \leq \mathbb{P}(E_1 \mid E_2) \leq 1$. [4]

2. (a) Suppose X and Y are jointly discrete random variables taking the values 0, 1 or 2 only, and that the joint probability mass function $p_{X,Y}(n,m)$ takes the values indicated in the table below:

		n		
	$p_{X,Y}(n,m)$	0	1	2
	0	0.16	0.12	0.12
m	1	0.12	0.09	0.09
	2	0.12	0.09	c

(i) Find the value of the constant c. [2]

(ii) Find:

α) $\mathbb{P}(X = Y)$ [2]

β) $\mathbb{P}(XY = 0)$ [2]

γ) $\mathbb{P}(X + Y \leq 2)$. [2]

(iii) Find the marginal probability mass function for Y and hence find $\mathbb{P}(Y > 0)$. [4]

(iv) Find $\mathbb{E}(XY)$. [2]

(b) Let U and V be independent jointly continuous random variables with identical marginal density function $f(x)$, $-\infty < x < \infty$. Use integration-by-parts to prove that

$$\mathbb{P}(U < V) = \frac{1}{2}.$$

Why is this result intuitive? [6]

3. (a) If X has moment-generating function $M_X(t)$ show that the moment-generating function for the random variable $aX + b$, with a, b constants is $\mathrm{e}^{bt} M_X(at)$. [4]

(b) A random variable X has an exponential distribution with parameter $\lambda > 0$ if it has probability density function

$$f_X(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the moment-generating function of X is

$$\frac{\lambda}{\lambda - t}, \qquad t < \lambda.$$ [5]

(c) Use your answer to part (b) to find the mean and variance of an exponential-distributed random variable. [5]

(d) Let $k > 0$ be a constant, and let $Y = kX$, where X is an exponential random variable with parameter $\lambda > 0$.

(i) Find the mean and variance of Y. [2]

(ii) Use uniqueness of moment-generating functions to prove that Y is an exponential random variable with parameter λ/k. [4]

4. (a) A pair of jointly continuous random variables have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 6\mathrm{e}^{-x-2y}, & 0 < x < y, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find a conditional density function $f_{X|Y}(x|y)$. [4]

(ii) Using the result of part (i) find the conditional expectation $\mathbb{E}(X \mid Y)$. [6]

(b) A Mathematics professor has three hats: a trilby, a Homburg and a pork-pie hat. If he wears the trilby one day, the next day he wears it again with probability $\frac{1}{3}$ and the pork-pie hat with probability $\frac{2}{3}$; if he wears the Homburg, he wears it again the next day with probability $\frac{1}{2}$ and the trilby with probability $\frac{1}{2}$; if he wears the pork-pie hat, he wears it again the next day with probability $\frac{1}{5}$ and the Homburg with probability $\frac{4}{5}$.

(i) Set this situation up as a Markov chain. What is the matrix of transition probabilities? [3]

(ii) If he wears the trilby on Monday, what is the probability he wears the Homburg on Wednesday? [3]

(iii) Which hat is worn more often over a long period of time? [4]