ANALYTICAL DYNAMICS
(22MAB255)

Semester 2 22/23 In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

Answer **3** questions.

You may assume any formula in the yellow formula book.

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1. (a) A thin uniform plate of mass M is a quarter of a circular disc of radius a (see Fig. Q1a).

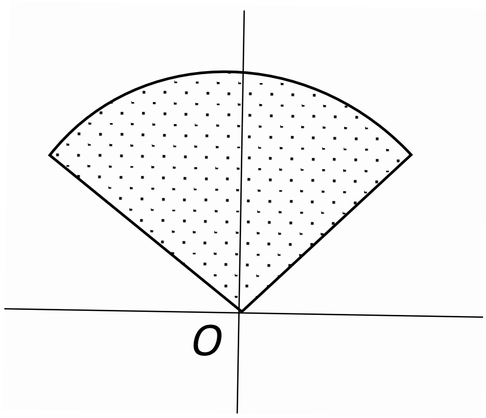

Fig. Q1a

(i) Find the distance from the centre of the disc O to the centre of mass of this plate. [4]

(ii) Find the moment of inertia of this plate about its axis of symmetry. [4]

(iii) Find the moment of inertia of this plate about the axis in its plane through the point O perpendicular to the axis of symmetry (see Fig. Q1a). [4]

(iv) Find the moment of inertia of this plate about the axis in its plane through the centre of mass perpendicular to the axis of symmetry. [2]

(b) A uniform solid right circular cone has height h, radius of the base a, and mass M. Show that the moment of inertia of this cone about its axis of symmetry is $3Ma^2/10$. [6]

2. A rectilinear tube can rotate freely around the vertical axis through the middle of the tube at a constant angle $\alpha \neq 0$ with the tube. The tube does not move up or down. A bead slides freely along this tube. The moment of inertia of the tube about the axis of rotation is I. The length of the tube is $2a$. The mass of the bead is m. The acceleration due to gravity is g. All constraints are ideal. No external forces other than gravity act on the system. The setup is illustrated in Fig. Q2.

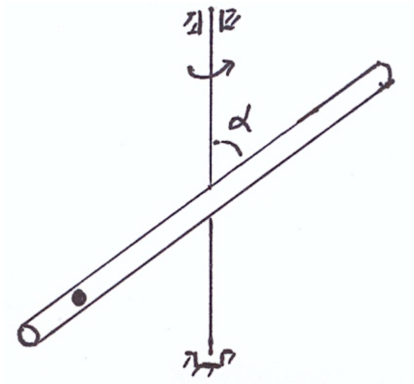

Fig. Q2

(a) Explain why the following two quantities are conserved during the motion: the angular momentum of the system about the axis of rotation L_z, and the total energy of the system E. [4]

(b) At an initial moment of time $t = 0$ the tube rotates with an angular speed ω_0, while the bead starts to move from the lower end of the tube towards the middle of the tube with a speed v_0 with respect to the tube.

(i) Find the angular speed of the tube when the bead is at the distance l from the middle of the tube. [4]

(ii) Find the speed of the bead with respect to the tube when the bead is at the distance l from the middle of the tube provided that the bead is below the middle of the tube. [7]

(iii) Prove that if $v_0 < \sqrt{2ga\cos\alpha}$ then the bead would not reach the middle of the tube. [5]

3. Consider a spherical pendulum with mass m of the bob and length l of the massless rod (see Fig. Q3). The acceleration due to gravity is g. The constraints are ideal.

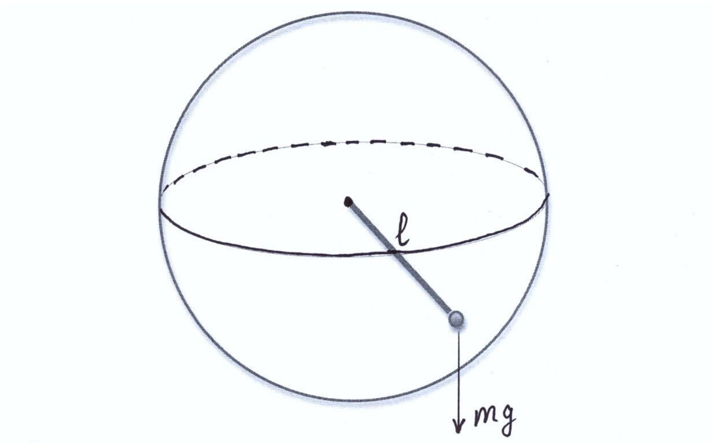

Fig. Q3

(a) How many degrees of freedom does this system have? Describe a possible system of generalised coordinates for this system. [3]

(b) Find the kinetic energy and the potential energy of the system. [4]

(c) Find the Lagrangian for the system. Write down Lagrange's equations of motion. [3]

(d) Find the Hamiltonian for the system. Write down Hamilton's equations of motion. [4]

(e) Write down two conservation laws for the spherical pendulum. [3]

(f) Let at some moment of time the pendulum pass through the upper vertical position. Show that the trajectory of the bob is a circle. [3]

4. A system with two degrees of freedom has the kinetic energy T and the potential energy V :

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2}m_2\left(l^2\dot{\varphi}^2 + 2l\dot{x}\dot{\varphi}\cos\varphi\right), \ V = -m_2 gl\cos\varphi + \frac{1}{2}kx^2 \,.$$

Here x and φ are generalised coordinates, and m_1, m_2, l, g, k are positive constants. The generalised coordinate φ is angular, i.e. values of φ that differ by 2π correspond to the same position of the system.

(a) Show that $x = 0, \varphi = 0$ is an equilibrium position of the system and find the other equilibrium position. Which of these equilibrium positions are stable? Justify your answer. [6]

(b) Find the equilibrium position having the lowest value of the potential energy. Write down the linearised Lagrange equations for the system in a small neighbourhood of this position. [6]

(c) Suppose that $m_1 = m_2 = m$, $k = mg/l$.

Find the eigenfrequencies of small oscillations near the equilibrium position with the lowest value of the potential energy. [8]