Loughborough University

ADVANCED COMPLEX ANALYSIS
(22MAC241)

Semester 2 22/23 In-Person Exam paper

This is an online long-window examination, meaning you have **23 hours** in which to complete and submit this paper. How you manage your time within the 23-hour window is up to you, but we expect you should only need to spend approximately **X hours** working on it. If you have extra time or rest breaks as part of a Reasonable Adjustment, you will need to add this to the amount of time you are expected to spend on the paper.

It is your responsibility to submit your work by the deadline for this examination. You must make sure you leave yourself enough time to do so.

It is also your responsibility to check that you have submitted the correct file.

Exam Help

If you are experiencing difficulties in accessing or uploading files during the exam period, you should contact the Exam Helpline. For urgent queries please call **01509 222900**.
For other queries email examhelp@lboro.ac.uk

You may handwrite and/or word process your answers, as you see fit.

You may use a calculator for this exam.

Answer **ALL FOUR** questions.

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1. (a) Does there exist an entire function $f(z)$ such that:

   (i) $f(x) = \mathrm{e}^x$ for all real x and $f(iy) = \cos y + \mathrm{i}\sin y$ for all real y. [5]

   (ii) $f(x) = \mathrm{e}^x$ for all real x and $f(iy) = \cos y - \mathrm{i}\sin y$ for all real y. [5]

   In each case, give a proof.

   (b) Show that if f is an entire function such that $f(2^n) = \mathrm{i}$ for all integer n, then:

   (i) $f'(0) = 0$. [5]

   (ii) $f^{(k)}(0) = 0$ for all $k \geq 1$. [5]

   (iii) $f(z) = \mathrm{i}$ for all $z \in \mathbb{C}$. [5]

2. For each function $f(z)$ below state with justification whether $f(z)$ has a zero, a pole or neither at $z = 0$. For zeros and poles, find their order.

   (a) $f(z) = z - \sin z$. [4]

   (b) $f(z) = z \sin z$. [7]

   (c) $f(z) = \dfrac{1}{z} \sin z$. [7]

   (d) $f(z) = z \sin \dfrac{1}{z}$. [7]

3. Compute:

   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4 + 1}\, dx$. [10]

   (b) $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}\ln x}{x^2 + 1}\, dx$. [15]

4. The Digamma function is defined by

$$\psi(z) = \frac{1}{\Gamma(z)} \frac{\mathrm{d}}{\mathrm{d}z}(\Gamma(z)).$$

In this question you may use facts you know about the gamma function $\Gamma(z)$ provided you state them clearly.

(i) Prove that $\psi(z)$ is a meromorphic function with simple poles at $z = 0, -1, -2, \ldots$ [6]

(ii) Show that

$$\psi(z) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \ln t \; \mathrm{e}^{-t} \; \mathrm{d}t \qquad \text{if} \qquad \mathfrak{Re}(z) > 0.$$ [6]

(iii) Show that $\psi(z)$, for $\mathfrak{Re}(z) > 0$, satisfies the functional equation

$$\psi(z+1) = \frac{1}{z} + \psi(z),$$

and hence show by induction that

$$\psi(z+n) = \sum_{j=0}^{n-1} \frac{1}{z+j} + \psi(z), \qquad n \in \mathbb{N}.$$ [7]

(iv) Show that

$$\sum_{n=0}^{99} \frac{1}{2n^2 + 7n + 3} = \frac{1}{5} \left(\psi\left(\frac{201}{2}\right) + \psi(3) - \psi(103) - \psi\left(\frac{1}{2}\right) \right).$$ [6]