ROBOTICS PLANNING AND CONTROL

22WSB009

Semester 2 2023 In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are **not** allowed).

ROBOTICS PLANNING AND CONTROL
(22WSB009)

Semester 2 2023 2 Hours

Answer **ALL** questions.

Questions carry the marks shown.

The total marks available for this paper is 60.

Any University-approved calculator is permitted.

A range of formulae and tables likely to be of benefit in the solution of these questions are provided at the rear of the paper.

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1. The dynamic model for the one degree of freedom robot arm in **Figure Q1-1** can be defined the mass matrix $M(\Theta) = [\,{}^{c}I_{zz} + mr_c^2]$, the velocity terms $V(\Theta,\, \dot{\Theta}) = [0]$, the gravity terms $G(\Theta) = [mr_c g\cos\theta]$, and the friction terms $F(\Theta,\, \dot{\Theta}) = [v\dot{\theta} + c\,\mathrm{sgn}(\dot{\theta})]$. Where the mass moment of inertia $I_{zz} = \frac{m}{12}(l^2 + w^2)$, the mass $m = 2kg$, the arm length $l = 1.0m$, the width of the arm $w = 0.05m$, the centre of mass location $r_c = 0.4m$, the viscous friction constant $\upsilon = 0.3Nms/rad$, and the Coulomb friction constant $c = 3Nm$. Assume that the gravity constant is $g = 9.81m/s^2$. The robot has unmodeled resonances at $\omega_{res} = 10.0$, 13.0, and $25.0rad/s$.

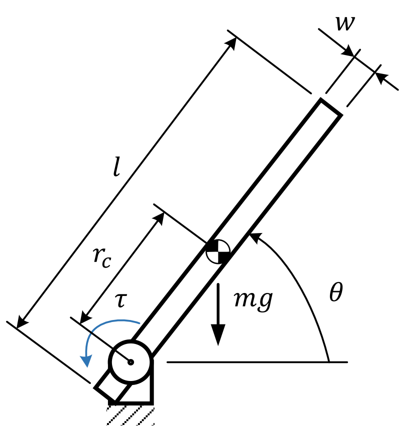

Figure Q1-1: Schematic diagram of robot arm

a) For the above robot (**Figure Q1-1**), draw a block diagram showing a trajectory following partitioned PD controller for the torque τ of a single join with disturbances τ_{dist}. Assume that both the pose θ and the velocity $\dot{\theta}$ of the joint are measured and that the desired trajectory is defined in position θ_d, velocity $\dot{\theta}_d$, and acceleration $\ddot{\theta}_d$ terms. Show the specific equations for this robot arm inside the blocks of the block diagram. [5 marks]

b) Determine the PD control gains, k_v and k_p, so that the system is always critically damped and as stiff as possible. Assume that the natural frequency of the closed-loop dynamics should be no larger than half the lowest resonance frequency of the robot arm, $\omega_n \leq \frac{1}{2}\omega_{res}$. [2 marks]

c) Determine the steady state pose error θ_e for partitioned PD control law if a constant disturbance of $\tau_{dist} = 4Nm$ is acting on the robot joint. [4 marks]

d) Explain different strategies by which the steady-state error in a closed-loop control system with disturbance can be reduced or even eliminated. Consider their implications for the control behaviour of the system. [4 marks]

2. You are tasked with designing the software architecture for a planner 3R robotic manipulator (see **Figure Q2-1**) using ROS. The robot is to be used in a manufacturing plant to assemble small electronic devices. The robot has an end-effector (EE) to pick and place components and a camera (Cam1) mounted on its EE. The manufacturing plant has a conveyor belt that brings in the components to be assembled, and a camera mounted (Cam2) on the ceiling for inspection.

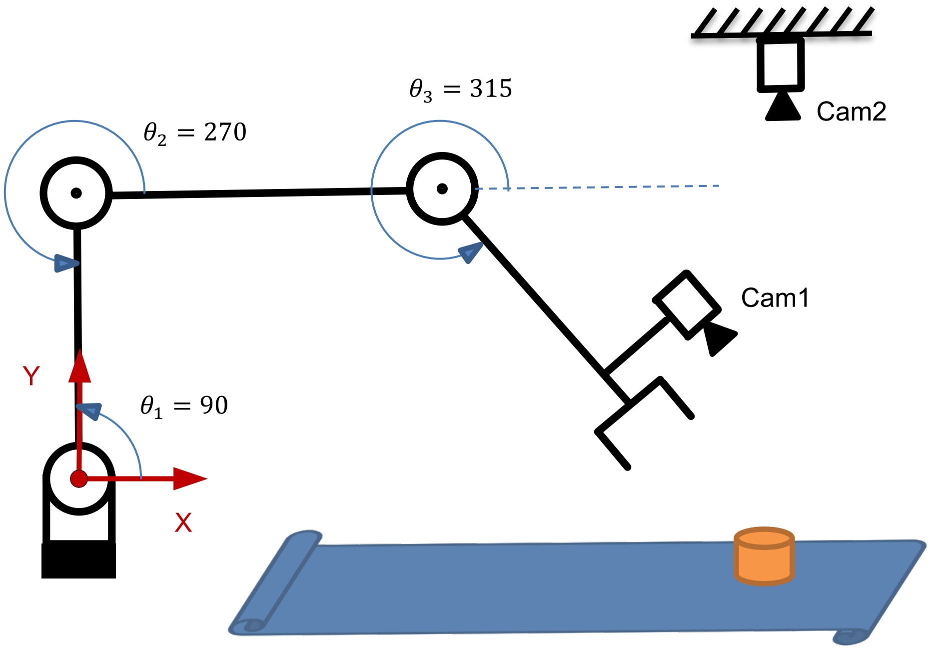

Figure Q2-1: Robot system schematic overview

a) The company would like to move the robot by teleoperate command. Describe the way ROS handles the message communications structure which could be used to parse this. Discuss the different ways that the nodes can communicate in ROS. [3 marks]

b) Discuss the types of sensors that could be used to improve the perception of the 3R robot and how this would help the visual inspection and pick and place tasks. Also discuss how the performance of these sensors can be assessed. Describe the process by which the robot is aware of its surroundings. [5 marks]

c) The camera mounted above the conveyor (Cam2) has a focal length of 50 mm and an image sensor of size 24 mm x 36 mm. The centre point of the component on the conveyer has the coordinates (2, 3, 4)m with respect to Cam2 coordinate system. Calculate the image coordinates of this point on the image sensor. Use a pinhole camera model. [2 marks]

d) The robot is tasked with grasping a component located on a conveyor in front of it. Cam1 provides a 2D image of the scene, and the pose of the component in the image is known. The transformation of the camera (Cam1) with respect to the EE (length is given in meters) and the transformation of the EE with respect to the Base, are given by the following matrices ($l_1 = l_2 = l_3 = 0.30$m):

$$ {}^{EE}_{Cam1}T = \begin{bmatrix} -1 & 0 & 0 & -0.05 \\ 0 & 0 & 1 & -0.10 \\ 0 & 1 & 0 & 0.00 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{Base}_{EE}T = \begin{bmatrix} cos(\theta_1 + \theta_2 + \theta_3) & -sin(\theta_1 + \theta_2 + \theta_3) & 0 & x \\ sin(\theta_1 + \theta_2 + \theta_3) & cos(\theta_1 + \theta_2 + \theta_3) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ x = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) $$

$$ y = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3) $$

Assuming that the robot arm is in its home position (θ_1=90°, θ_2=270° and θ_3=315°), determine the 3D pose (position and orientation) of the component with respect to the robot's base frame, given the known 2D pose of the component in the image and the intrinsic parameters of the camera.

2D pose of the component in the image: $p = \begin{bmatrix} 120 \\ 150 \end{bmatrix}$

Intrinsic matrix: $A = \begin{bmatrix} 1394 & 0 & 995 \\ 0 & 1394 & 600 \\ 0 & 0 & 1 \end{bmatrix}$

Assume: λ=1 [10 marks]

3. XPTO Technologies ltd provides consultancy services for robotic solutions. The company was approach by a customer which provided the configuration space seen in **Figure Q3-1**.

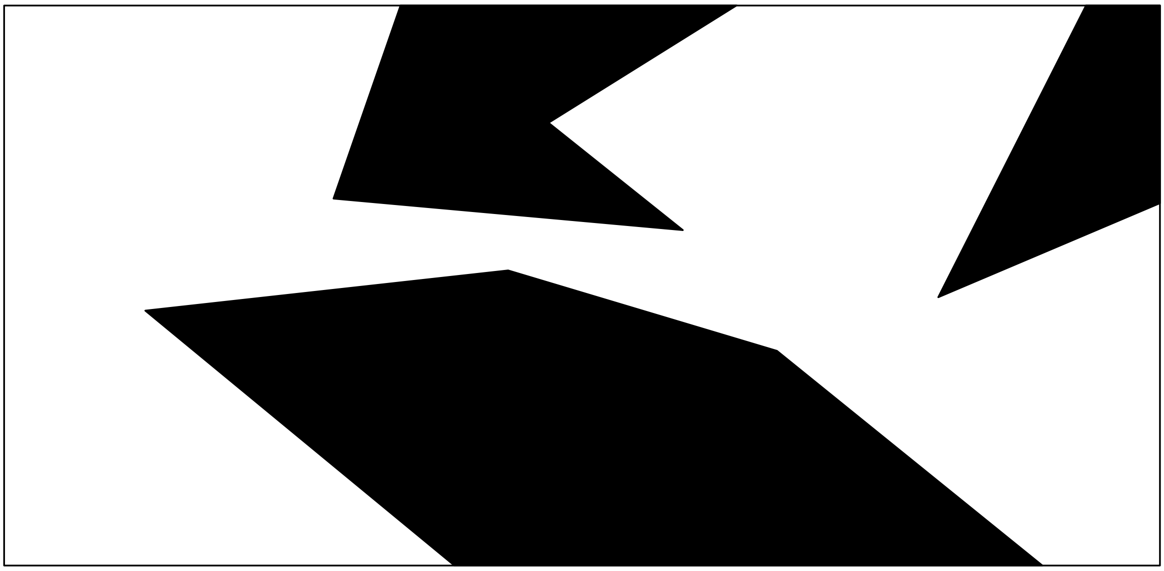

Figure Q3-1: Configuration space.

a) The customer requested an Exact Cell Decomposition of **Figure Q3-1**, including the generation of the resulting graph. [4 marks]

b) The customer provides the sketch of different graph which includes weights to the possible paths and an area-based heuristic (h), as seen in **Figure Q3-2**.

   i. Starting from A to reach J, please provide the final tree view of applying Dijkstra algorithm and highlight the selected path. [8 marks]

   ii. Starting from A to reach J, please provide the final tree view of applying A* algorithm and highlight the selected path. [8 marks]

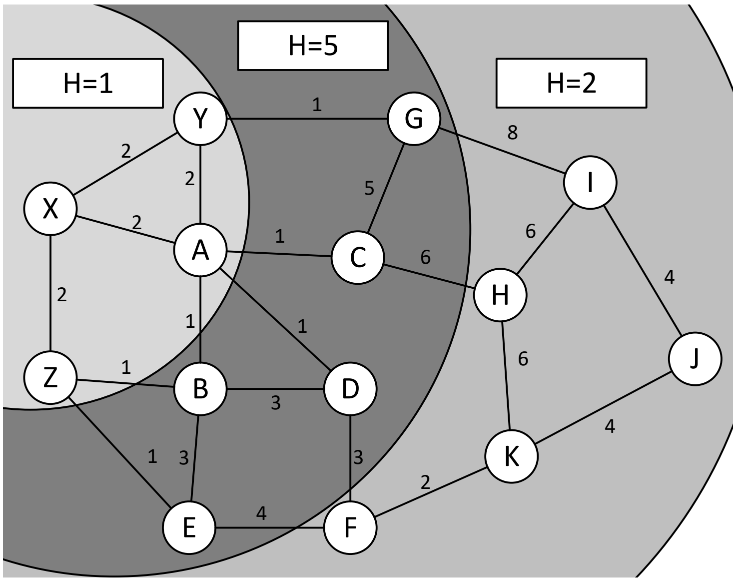

Figure Q3-2: Graph for paths with weights and heuristic.

c) The customer started a Rapidly Exploring Random Trees (RRTs) algorithm process as seen in **Figure Q3-3**. Consider the points 1 to 5 sequentially as the random point in the algorithm. Establish the resulting RTTs graph with these points and considering the indicated segment max length. [5 marks]

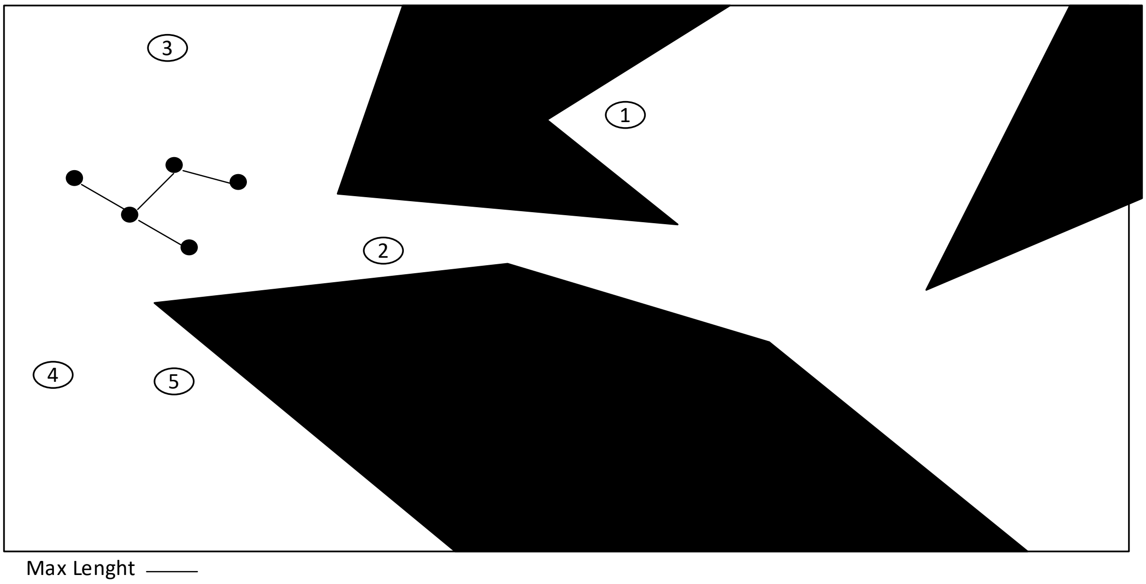

Figure Q3-3: Configuration space with RRT with four branches.

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Formula Collection

General

Vector Addition

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$$

Vector Subtraction

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} - \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_x - b_x \\ a_y - b_y \\ a_z - b_z \end{bmatrix}$$

Vector Dot Product

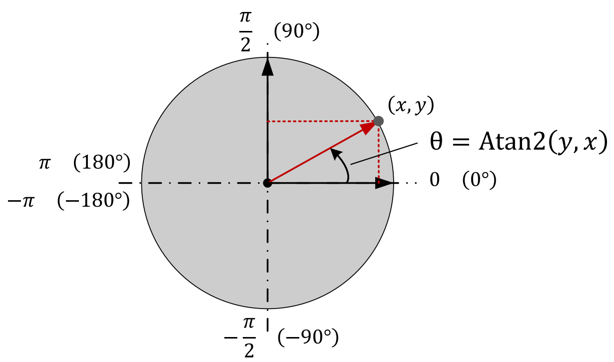
$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \cdot \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = a_x \times b_x + a_y \times b_y + a_z \times b_z$$

Vector Cross Product

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \times \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}$$

Atan2 Function

$$\mathrm{Atan2}(y,x) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$



Position and Orientation

Description of Position

$${}^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

/continued

Description of Orientation

$$
{}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

Elementary Rotations

$$
{}^{A}_{B}R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}
$$

$$
{}^{A}_{B}R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
$$

$$
{}^{A}_{B}R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

X-Y-Z Fixed Angle Representation

$$
{}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} \mathrm{c}\alpha\mathrm{c}\beta & \mathrm{c}\alpha\mathrm{s}\beta\mathrm{s}\gamma - \mathrm{s}\alpha\mathrm{c}\gamma & \mathrm{c}\alpha\mathrm{s}\beta\mathrm{c}\gamma + \mathrm{s}\alpha\mathrm{s}\gamma \\ \mathrm{s}\alpha\mathrm{c}\beta & \mathrm{s}\alpha\mathrm{s}\beta\mathrm{s}\gamma + \mathrm{c}\alpha\mathrm{c}\gamma & \mathrm{s}\alpha\mathrm{s}\beta\mathrm{c}\gamma - \mathrm{c}\alpha\mathrm{s}\gamma \\ -\mathrm{s}\beta & \mathrm{c}\beta\mathrm{s}\gamma & \mathrm{c}\beta\mathrm{c}\gamma \end{bmatrix}
$$

Inverse:

$$
\beta = \operatorname{Atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)
$$

$$
\alpha = \operatorname{Atan2}\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right)
$$

$$
\gamma = \operatorname{Atan2}\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right)
$$

Z-Y-X Cardano Angle Representation

$$
{}^{A}_{B}R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} \mathrm{c}\alpha\mathrm{c}\beta & \mathrm{c}\alpha\mathrm{s}\beta\mathrm{s}\gamma - \mathrm{s}\alpha\mathrm{c}\gamma & \mathrm{c}\alpha\mathrm{s}\beta\mathrm{c}\gamma + \mathrm{s}\alpha\mathrm{s}\gamma \\ \mathrm{s}\alpha\mathrm{c}\beta & \mathrm{s}\alpha\mathrm{s}\beta\mathrm{s}\gamma + \mathrm{c}\alpha\mathrm{c}\gamma & \mathrm{s}\alpha\mathrm{s}\beta\mathrm{c}\gamma - \mathrm{c}\alpha\mathrm{s}\gamma \\ -\mathrm{s}\beta & \mathrm{c}\beta\mathrm{s}\gamma & \mathrm{c}\beta\mathrm{c}\gamma \end{bmatrix}
$$

/continued

Z-Y-Z Euler Angle Representation

$$^{A}_{B}R_{Z'Y'Z'}(\alpha, \beta,\ \gamma) = \begin{bmatrix} \mathrm{c}\alpha\mathrm{c}\beta\mathrm{c}\gamma - \mathrm{s}\alpha\mathrm{s}\gamma & -\mathrm{c}\alpha\mathrm{c}\beta\mathrm{s}\gamma - \mathrm{s}\alpha\mathrm{c}\gamma & \mathrm{c}\alpha\mathrm{s}\beta \\ \mathrm{s}\alpha\mathrm{c}\beta\mathrm{c}\gamma + \mathrm{c}\alpha\mathrm{s}\gamma & -\mathrm{s}\alpha\mathrm{c}\beta\mathrm{s}\gamma + \mathrm{c}\alpha\mathrm{c}\gamma & \mathrm{s}\alpha\mathrm{s}\beta \\ -\mathrm{s}\beta\mathrm{c}\gamma & \mathrm{s}\beta\mathrm{s}\gamma & \mathrm{c}\beta \end{bmatrix}$$

Inverse:

$$\beta = \mathrm{Atan2}\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

$$\alpha = \mathrm{Atan2}\left(\frac{r_{23}}{\sin\beta}, \frac{r_{13}}{\sin\beta}\right)$$

$$\gamma = \mathrm{Atan2}\left(\frac{r_{32}}{\sin\beta}, -\frac{r_{31}}{\sin\beta}\right)$$

Equivalent Angle-Axis Representation

$$^{A}_{B}R_{K}(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

- Where $c\theta = \cos\theta$, $s\theta = \sin\theta$, $v\theta = 1 - \cos\theta$, and $^{A}\hat{K} = [k_x \quad k_y \quad k_z]^T$

Inverse:

$$\cos\theta = \frac{r_{11} + r_{22} + r_{33} - 1}{2}$$

$$^{A}\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Euler Parameters (Unit Quaternions) Representation $\boldsymbol{Q} = (\boldsymbol{\eta}, \boldsymbol{\epsilon})$

$$^{B}_{A}R(\eta, \epsilon) = \begin{bmatrix} 1 - 2\epsilon_y^2 - 2\epsilon_z^2 & 2(\epsilon_x\epsilon_y - \epsilon_z\eta) & 2(\epsilon_x\epsilon_z + \epsilon_y\eta) \\ 2(\epsilon_x\epsilon_y + \epsilon_z\eta) & 1 - 2\epsilon_x^2 - 2\epsilon_z^2 & 2(\epsilon_y\epsilon_z - \epsilon_x\eta) \\ 2(\epsilon_x\epsilon_z - \epsilon_y\eta) & 2(\epsilon_y\epsilon_z + \epsilon_x\eta) & 1 - 2\epsilon_x^2 - 2\epsilon_y^2 \end{bmatrix}$$

- Where $\epsilon = [\epsilon_x \quad \epsilon_y \quad \epsilon_z]^T$, $\hat{K} = [k_x \quad k_y \quad k_z]^T$, $\epsilon = \hat{K}\sin\frac{\theta}{2}$, and $\eta = \cos\frac{\theta}{2}$
- Subject to: $\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1$

/continued

Inverse:

$$\eta = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$\epsilon_x = \frac{r_{32} - r_{23}}{4\eta}$$

$$\epsilon_y = \frac{r_{13} - r_{31}}{4\eta}$$

$$\epsilon_z = \frac{r_{21} - r_{12}}{4\eta}$$

Manipulator Kinematics

DH Transformation

$${}^{i-1}_{i}T = R_X(\alpha_{i-1})D_X(a_{i-1})R_Z(\theta_i)D_Z(d_i)$$

$${}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Where:
  - a_i: is the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$
  - α_i: is the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$
  - d_i: is the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$
  - θ_i: is the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$

Manipulator Instantaneous Velocities

Angular Velocity

Revolute Joints

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

Prismatic Joints

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_i$$

Linear Velocity

Revolute Joints

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right)$$

Prismatic Joints

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right) + \dot{d}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}$$

Jacobians in the Velocity Domain

$$^{0}\nu = {}^{0}J(\Theta)\dot{\Theta}$$

Changing a Jacobian's Frame of Reference

$$^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)$$

Cartesian Transformation of Velocities

$$^{A}\nu_A = {}^{A}_{B}T_{\upsilon}\, {}^{B}\nu_B$$

$$\begin{bmatrix} {}^{A}v_A \\ {}^{A}\omega_A \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B.org} \times {}^{A}_{B}R \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v_B \\ {}^{B}\omega_B \end{bmatrix}$$

Static Forces in Manipulators

Static Forces Balance in Link i

$$^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$$

Static Moments Balance in Link i

$$^{i}n_i = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$

Static Joint Torques

Revolute Joints

$$\tau_i = {}^{i}n_i^T\, {}^{i}\hat{Z}_i$$

Prismatic Joints

$$\tau_i = {}^{i}f_i^T\, {}^{i}\hat{Z}_i$$

/continued

Jacobians in the Force Domain

$\tau = {}^{0}J^{T}\ {}^{0}\mathcal{F}$

Cartesian Transformation of Static Forces

${}^{A}\mathcal{F}_{A} = {}^{A}_{B}T_{f}\ {}^{B}\mathcal{F}_{B}$

$$\begin{bmatrix} {}^{A}F_{A} \\ {}^{A}N_{A} \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ {}^{A}P_{B.org} \times {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}F_{B} \\ {}^{B}N_{B} \end{bmatrix}$$

Manipulator Dynamics

Newton-Euler Dynamic Formulation

Rotational Velocity

Revolute Joints

${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$

Prismatic Joints

${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega_{i}$

Angular Acceleration

Revolute Joints

${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R\ {}^{i}\omega_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$

Prismatic Joints

${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}_{i}$

Linear Acceleration

Revolute Joints

${}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) + {}^{i}\dot{v}_{i} \right)$

Prismatic Joints

$$\begin{aligned} {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times \left({}^{i}\omega_{i} \times {}^{i}P_{i+1} \right) + {}^{i}\dot{v}_{i} \right) + 2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \\ + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \end{aligned}$$

Linear acceleration of the centre of mass

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}$$

Force action on the centre of mass

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

Torque action on the centre of mass

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

Link force

$$^{i}f_i = {}_{i+1}^{\;\;i}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

Link torque

$$^{i}n_i = {}^{i}N_i + {}_{i+1}^{\;\;i}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}_{i+1}^{\;\;i}R\,{}^{i+1}f_{i+1}$$

Joint torque

Revolute Joints

$$\tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i$$

Prismatic Joints

$$\tau_i = {}^{i}f_i^T\,{}^{i}\hat{Z}_i$$

 /continued

State-Space Equation

Joint Space

$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$

- Where:
  - $M(\Theta)$: is the n x n mass matrix of the manipulator
  - $V(\Theta, \dot{\Theta})$: is an n x 1 vector of centrifugal and Coriolis terms
  - $G(\Theta)$: is an n x 1 vector of gravity terms

Including Friction

$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$

Cartesian Space

$\mathcal{F} = M_x(\Theta)\ddot{\chi} + V_x(\Theta, \dot{\Theta}) + G_x(\Theta)$

- Where:
  - $M_x(\Theta) = J^{-T}(\Theta)M(\Theta)J^{-1}(\Theta)$
  - $V_x(\Theta, \dot{\Theta}) = J^{-T}(\Theta)\left(V(\Theta, \dot{\Theta}) - M(\Theta)J^{-1}(\Theta)\dot{J}(\Theta)\dot{\Theta}\right)$
  - $G_x(\Theta) = J^{-T}(\Theta)G(\Theta)$

Configuration-Space Equation

Joint Space

$\tau = M(\Theta)\ddot{\Theta} + B(\Theta)[\dot{\Theta}\dot{\Theta}] + C(\Theta)[\dot{\Theta}^2] + G(\Theta)$

- Where:
  - $B(\Theta)$: is a matrix of $n \times n(n-1)/2$ dimensions of Coriolis coefficients
  - $[\dot{\Theta}\dot{\Theta}]$: is a $n(n-1)/2 \times 1$ vector of joint velocity products given by:
    $$[\dot{\Theta}\dot{\Theta}] = [\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \dots \quad \dot{\theta}_{n-1}\dot{\theta}_n]^T$$
  - $C(\Theta)$: is an $n \times n$ matrix of centrifugal coefficients
  - $[\dot{\Theta}^2]$: is an $n \times 1$ vector given by:
    $$[\dot{\Theta}^2] = [\dot{\theta}_1^2 \quad \dot{\theta}_2^2 \quad \dots \quad \dot{\theta}_n^2]^T$$

/continued

Cartesian Space

$$\tau = J^T(\Theta)M_x(\Theta)\ddot{X} + B_x(\Theta)\left[\dot{\Theta}\dot{\Theta}\right] + C_x(\Theta)\left[\dot{\Theta}^2\right] + G(\Theta)$$

- Where:
  - $B_x(\Theta)$: is a matrix of $n \times n(n-1)/2$ dimensions of Coriolis coefficients
  - $\left[\dot{\Theta}\dot{\Theta}\right]$: is a $n(n-1)/2 \times 1$ vector of joint velocity products given by:
    $$\left[\dot{\Theta}\dot{\Theta}\right] = [\dot{\theta}_1\dot{\theta}_2 \quad \dot{\theta}_1\dot{\theta}_3 \quad \dots \quad \dot{\theta}_{n-1}\dot{\theta}_n]^T$$
  - $C_x(\Theta)$: is an $n \times n$ matrix of centrifugal coefficients
  - $\left[\dot{\Theta}^2\right]$: is an $n \times 1$ vector given by:
    $$\left[\dot{\Theta}^2\right] = [\dot{\theta}_1^2 \quad \dot{\theta}_2^2 \quad \dots \quad \dot{\theta}_n^2]^T$$

Trajectory Generation

Cubic Polynomial

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

Coefficients for predefined positions and velocities:

- $a_0 = \theta_0$
- $\mathrm{a}_1 = \dot{\theta}_0$
- $a_2 = \frac{3}{t_f^2}\left(\theta_f - \theta_0\right) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$
- $a_3 = -\frac{2}{t_f^3}\left(\theta_f - \theta_0\right) + \frac{1}{t_f^2}\left(\dot{\theta}_f + \dot{\theta}_0\right)$

 /continued

Quintic Polynomial

$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$

$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$

$\ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$

Coefficients for predefined positions, velocities, and accelerations:

- $a_0 = \theta_0$
- $a_1 = \dot{\theta}_0$
- $a_2 = \frac{\ddot{\theta}_0}{2}$
- $a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f - (3\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^3}$
- $a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f + (3\ddot{\theta}_0 - 2\ddot{\theta}_f)t_f^2}{2t_f^4}$
- $a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f - (\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^5}$

Linear Function with Parabolic Blends

$\ddot{\theta}t_b^2 - \ddot{\theta}t t_b + (\theta_f - \theta_0) = 0$

Where:

- t_b: is the time at the end of the blend region
- θ_0: is the value of θ at $t = 0$
- θ_f: is the value of θ at the finish of the trajectory

Blend time for a chosen **acceleration**, $\ddot{\theta}_b$:

- $t_b = \frac{t_f}{2} - \frac{\sqrt{\ddot{\theta}_b^2 t_f^2 - 4\ddot{\theta}_b(\theta_f - \theta_0)}}{2\ddot{\theta}_b}$

Subject to:

- $|\ddot{\theta}_b| \geq \frac{4|\theta_f - \theta_0|}{t_f^2}$

Blend time and acceleration for a chosen **cruise velocity**, $\dot{\theta}_b$:

- $t_b = \frac{\theta_0 - \theta_f + \dot{\theta}_b t_f}{\dot{\theta}_b}$
- $\ddot{\theta}_b = \frac{\dot{\theta}_b^2}{\theta_0 - \theta_f + \dot{\theta}_b t_f}$

Subject to:

- $\frac{|\theta_f - \theta_0|}{t_f} < |\dot{\theta}_b| \leq \frac{2|\theta_f - \theta_0|}{t_f}$

Polynomial functions for position, velocity, and acceleration:

$$\theta(t)=\begin{cases}\theta_0+\frac{1}{2}\ddot{\theta}_b t^2 & \text{if } 0\le t\le t_b\\ \theta_0+\ddot{\theta}_b t_b\left(t-\frac{1}{2}t_b\right) & \text{if } t_b<t\le t_f-t_b\\ \theta_f-\frac{1}{2}\ddot{\theta}_b\left(t_f-t\right)^2 & \text{if } t_f-t_b<t\le t_f\end{cases}$$

$$\dot{\theta}(t)=\begin{cases}\ddot{\theta}_b t & \text{if } 0\le t\le t_b\\ \ddot{\theta}_b t_b & \text{if } t_b<t\le t_f-t_b\\ \ddot{\theta}_b\left(t_f-t\right) & \text{if } t_f-t_b<t\le t_f\end{cases}$$

$$\ddot{\theta}(t)=\begin{cases}\ddot{\theta}_b & \text{if } 0\le t\le t_b\\ 0 & \text{if } t_b<t\le t_f-t_b\\ -\ddot{\theta}_b & \text{if } t_f-t_b<t\le t_f\end{cases}$$

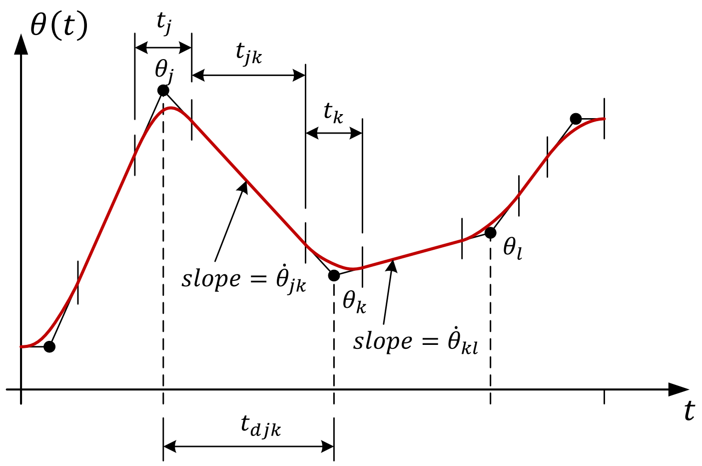
Linear Function with Parabolic Blends – with Via Points

$$\dot{\theta}_{jk}=\frac{\theta_k-\theta_j}{t_{djk}}$$

$$\ddot{\theta}_k=SGN\left(\dot{\theta}_{kl}-\dot{\theta}_{jk}\right)\left|\ddot{\theta}_k\right|$$

$$t_k=\frac{\dot{\theta}_{kl}-\dot{\theta}_{jk}}{\ddot{\theta}_k}$$

$$t_{jk}=t_{djk}-\frac{1}{2}t_j-\frac{1}{2}t_k$$



Where:

- t_k: is the duration of the blend region for path point k.
- t_{jk}: is the duration of the linear portion between j and k.
- t_{djk}: is the overall duration of the segment connecting j and k.
- $\dot{\theta}_{jk}$: is the velocity of the linear portion between j and k.
- $\ddot{\theta}_j$: is the acceleration during the blend at point j.

/continued

First segment:

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2}t_1} = \ddot{\theta}_1 t_1$$

$$t_1 = t_{d12} - \sqrt{t_{d12}^2 - \frac{2(\theta_2 - \theta_1)}{\ddot{\theta}_1}}$$

Where:

- $\ddot{\theta}_1 = SGN(\theta_2 - \theta_1)\left|\ddot{\theta}_1\right|$

Last segment:

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2}t_n} = \ddot{\theta}_n t_n$$

$$t_n = t_{d(n-1)n} - \sqrt{t_{d(n-1)n}^2 + \frac{2(\theta_n - \theta_{n-1})}{\ddot{\theta}_n}}$$

Where:

- $\ddot{\theta}_n = SGN(\theta_{n-1} - \theta_n)\left|\ddot{\theta}_n\right|$

Linear Control of Manipulators

Second-Order Linear System

$$\ddot{x}_e + 2\zeta\omega_n\dot{x}_e + \omega_n^2 x_e = 0$$

Where:

- x_e: is the position error between the desired and actual position $x_e = x_d - x$
- $\dot{x}_e$: is the velocity error
- $\ddot{x}_e$: is the acceleration error
- $\omega_n = \sqrt{\frac{k}{m}}$: is called the **natural frequency**.
- $\zeta = \frac{b}{2\sqrt{km}}$: is called the **damping ratio**.

The roots are:

- $s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$
- $s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

Overdamped: $\zeta > 1$

- The solution for $\ddot{x}_e + 2\zeta\omega_n\dot{x}_e + \omega_n^2 x_e = 0$ is:
  $x_e(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$
- The coefficients for standard step error response $x_e(0) = 1$ and $\dot{x}_e(0) = 0$, are:
  $c_1 = \frac{1}{2} + \frac{\zeta}{2\sqrt{\zeta^2 - 1}}$
  $c_2 = \frac{1}{2} - \frac{\zeta}{2\sqrt{\zeta^2 - 1}}$

Critically damped: $\zeta = 1$

- The solution for$\ddot{x}_e + 2\zeta\omega_n\dot{x}_e + \omega_n^2 x_e = 0$ is:
  $x_e(t) = (c_1 + c_2 t)e^{-\omega_n t}$
- The coefficients for standard step error response $x_e(0) = 1$ and $\dot{x}_e(0) = 0$, are:
  $c_1 = 1$
  $c_2 = \omega_n$

Underdamped: $\zeta < 1$

- The solution for $\ddot{x}_e + 2\zeta\omega_n\dot{x}_e + \omega_n^2 x_e = 0$ is:
  $x_e(t) = (c_1 \cos\omega_d t + c_2 \sin\omega_d t)e^{-\zeta\omega_n t}$
  Where:
  - $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is the **damped natural frequency**
- The coefficients for standard step error response $x_e(0) = 1$ and $\dot{x}_e(0) = 0$, are:
  $c_1 = 1$
  $c_2 = \frac{\zeta}{\sqrt{1 - \zeta^2}}$

Partitioned Control Law

Servo portion of the control law:

$f' = \ddot{x}_d + k_v\dot{x}_e + k_p x_e$

Where:

- $\ddot{x}_d$: is the desired acceleration
- k_v: is the velocity control gain
- k_p: is the position control gain

/continued

Model-based portion of the control law:

$f = \alpha f' + \beta$

Where:

- α and β are functions or constants that are chosen such that if f' is taken as the new input, ***the system appears to be a unit mass***.

Error Dynamics for Trajectory-Following:

$\ddot{x}_e + k_v \dot{x}_e + k_p x_e = 0$

Error Dynamics for Trajectory-Following with Disturbance rejection:

$\ddot{x}_e + k_v \dot{x}_e + k_p x_e = \frac{1}{\alpha} f_{dis}$

Where:

- f_{dis}: is a disturbance force

Partitioned Control Law of the Single Joint

Model of the joint torque:

$\tau = (I + \eta^2 I_m)\ddot{\theta} + (b + \eta^2 b_m)\dot{\theta}$

Where:

- I_m and I are the **inertias** of the motor rotor and the load, respectively.
- b_m and b are **viscous friction coefficients** for the rotor and load bearings, respectively.

Model-based portion of the control law:

$\tau = \alpha \tau' + \beta$

Servo portion of the control law:

$\tau' = \ddot{\theta}_d + k_v \dot{\theta}_e + k_p \theta_e$

Where the control gains are chosen as:

- $k_p = \omega_n^2 = \frac{1}{4}\omega_{res}^2$
- $k_v = 2\sqrt{k_p} = \omega_{res}$

With:

- ω_{res}: is the resonance frequency of the system

MIMO Partitioned Control of Manipulator

The **rigid body dynamics** of a robotic manipulator take the form:

$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$

Where:

- τ: is an $n \times 1$ vector of the joint torques/forces
- $M(\Theta)$: is the $n \times n$ inertia matrix
- $V(\Theta, \dot{\Theta})$: is an $n \times 1$ vector of centrifugal and Coriolis terms
- $G(\Theta)$: is an $n \times 1$ vector of gravity terms
- $F(\Theta, \dot{\Theta})$: is an $n \times 1$ vector of friction terms

The **model-based portion** of the control law is:

$\tau = \alpha\tau' + \beta$

Where:

- τ, τ', and β are $n \times 1$ vectors
- α is a $n \times n$ matrix

The **servo portion** of the control law is:

$\tau' = \ddot{\Theta}_d + K_v \dot{\Theta}_e + K_p \Theta_e$

Where:

- K_v and K_p are $n \times n$ matrices, generally chosen with constant gains on the diagonal.
- Θ_e and $\dot{\Theta}_e$ are $n \times 1$ vectors of the position and velocity errors, respectively.

Perception

Sensor performance

Dynamic Range (dB): $10\,.\log\left[\frac{\text{upper limit}}{\text{lower limit}}\right]$

$accuracy = 1 - \frac{|error|}{v}$

$precision = \frac{range}{\sigma}$

Probability of X falling between two limits *a* and *b*:

$$p[a < X \leq b] = \int_{a}^{b} f(x)dx$$

Mean Value: $\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$

Variance: $Var(X) = \sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

Gaussian function: $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\ exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

Time of Flight (TOF): $d = c.t$

Where:

- d = distance travelled,
- c = speed of wave propagation,
- t = time of flight

Ultrasonic TOF: d=($c.t$)/2

Where:

- d = distance calculated,
- c = speed of sound in air ($c = \sqrt{\gamma RT}$),
- t = time of flight,
- γ = Ratio of specific heats,
- R = Gas constant,
- T = Temperature in degrees Kelvin

Infrared Phase Shift Measurement:

$$D' = L + 2D = L + \frac{\theta}{2\pi}\lambda$$

$$c = f.\lambda$$

$$D_{required} = \frac{\theta}{4\pi}\lambda$$

Where:

- c = speed of light,
- f = modulating frequency,
- λ = the wavelength,
- θ = the phase difference,
- D' = Total distance covered by the emitted light,
- D = Distance between the half-silvered mirror and the mirror,
- L = Distance between the half-silvered mirror and the detector,
- $D_{required}$ = The required distance between the beam splitter and the target

Doppler Effect Sensing:

If the transmitter is moving: $f_r = f_t \frac{1}{1+v/c}$

If the receiver is moving: $f_r = f_t\ (1 + v/c)$

Where:

- v = the relative speed between the transmitter and receiver,
- f_r = frequency of receiver electromagnetic wave,
- f_t = frequency of transmitter electromagnetic wave

Camera pinhole

Basic Lens model: $\frac{1}{f} = \frac{1}{z} + \frac{1}{e}$

Where:

- z = distance to the object,
- e = distance behind the lens at which the focused image is formed,
- f = focal length

Blur circle: $R = \frac{L\delta}{2e}$

Where:

- R = Radius of Circle,
- L = the diameter of the Lens,
- δ = the displacement of the image plane

Simple Perspective Projection: $\frac{f}{z} = \frac{u}{x} = \frac{v}{y}$

Where:

- $P_W(x, y, z)$ = position of the point in the world frame and
- $p(u, v)$ = position of the point in the image frame
- f = focal length

General camera model

$$u = k_u \frac{f}{z}.x + c_x$$

$$v = k_v \frac{f}{z}.y + c_y$$

Where:

- (c_x, c_y) are the coordinates of the principle point,
- $k_u(k_v)$ is the inverse of the effective pixel size along u (v) direction and is measured in pixel.m^-1.

$$p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \, and \; P_w = \begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix}$$

Where:

- $P_W(x, y, z)$ = position of the point in the world frame and
- $p(u, v)$ = position of the point in the image frame

Camera model:

$$\lambda p = A[R|t]P_w$$

$$\text{intrinsic matrix: } A = \begin{bmatrix} f_x & 0 & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

camera extrinsic parameters: $f_x = fk_u$ and $f_y = fk_v$ which describe the focal lengths expressed in horizontal and vertical pixels respectively

The Rotational and Translational parameters *R* and *t,* respectively.

Line fitting

Probabilistic line fitting: $p_i \cos(\theta_i - \alpha) - r = d_i$

i^{th} Point in polar coordinates: $x_i(p_i,\theta_i)$

Where:

- d_i : the orthogonal distance between (p_i,θ_i) and the line
- r and α: Model parameters in polar coordinates

Individual weight for Probabilistic line fitting: $w_i = 1/{\sigma_i}^2$

Where:

- σ^2 is Variance

Cost function: $S=\sum_i w_i {d_i}^2 = \sum_i w_i(p_i \cos(\theta_i - \alpha) - r)^2$

The weighted least-squares solution:

$$\alpha=\frac{1}{2}atan\left(\frac{\sum_i w_i {p_i}^2\ sin2\theta_i - \frac{2}{\sum_i w_i}\sum\sum w_i w_j p_i p_j \cos\theta_i sin\theta_j}{\sum_i w_i {p_i}^2\ cos2\theta_i - \frac{1}{\sum_i w_i}\sum\sum w_i w_j p_i p_j \cos(\theta_i + \theta_j)}\right)$$

$$r=\frac{\sum_i w_i p_i \cos(\theta_i - \alpha)}{\sum_i w_i}$$

Error Propagation

Covariance matrix: $C_Y = F_X C_X {F_X}^T$

Where:

- C_X = covariance matrix representing the input uncertainties
- C_Y = covariance matrix representing the propagated uncertainties for the outputs
- F_X is the Jacobian matrix defined as: $F_X = \nabla f = \begin{bmatrix} \frac{df_1}{dX_1} & \cdots & \frac{df_1}{dX_n} \\ \vdots & \ddots & \vdots \\ \frac{df_m}{dX_1} & \cdots & \frac{df_m}{dX_n} \end{bmatrix}$