COMPUTATIONAL FLUID DYNAMICS 1

(22WSC802)

Semester 2 2023 | 2 Hours

Answer **ALL THREE** questions.

All questions carry **EQUAL** marks.

You may take **TWO A4** sides of your own notes into the examination venue.

Any calculator is permitted.

A set of useful equations is attached at the end of this paper.

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1. **Figure Q.1** shows the cross-section of a small vessel to study the flow of a fluid in the annular gap between two concentric cylinders one of which is rotating. The dimensions are as follows: inside radius R_i = 45 mm and outside radius R_o = 60 mm. The outer cylinder is stationary and the inner cylinder is rotated around its axis at ω = 30 rad/s. The steady flow in the annular gap is governed by the following equation.

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d(rU_\theta)}{dr}\right) = 0$$

The equation describes the variation of the circumferential fluid velocity U_θ as a function of the radius r.

a) Use the finite volume method and central differencing to derive the discretised form of the equation for the three nodes of the uniform grid shown in **Figure Q.1**. [12 marks]

b) Solve the discretised equations to determine estimates of the circumferential velocity U_θ at nodes 1, 2 and 3. [4 marks]

c) Calculate the values of the circumferential velocity U_θ at the nodal points using the analytical solution $U_\theta(r) = \omega R_i \times \left(\frac{\frac{R_o}{r} - \frac{r}{R_o}}{\frac{R_o}{R_i} - \frac{R_i}{R_o}}\right)$ and determine the error of the numerical solution.

[4 marks]

2. The staggered grid shown in **Figure Q.2** may be used to calculate one-dimensional flow through the converging duct shown. The cross-sectional areas (in m^2) at different sections of the nozzle are indicated in **Figure Q.2**. The velocities are calculated at locations A, B, C, D and E while the pressure p is calculated at node locations 1, 2, 3, 4 and 5. In the solution algorithm a guessed pressure field $p*$ is used to solve the relevant momentum equation to give an intermediate velocity field $u*$. The correct velocity is obtained from:

$$u_i = u_i^* + d_i\left(p'_{I-1} - p'_I\right)$$

where locations $I-1$ and I lie on either side of the *u*-velocity node and p' stands for pressure correction. At an intermediate stage of the calculation u_B^* = 2.0 m/s, u_C^* = 2.0 m/s , u_D^* = 2.0 m/s and u_E^* = 2.0 m/s. Here $d=1$ and density $\rho = 1$ kg/m^3 everywhere and the boundary conditions are u_A = 2.0 m/s and p'_5 = 0.

a) Use the methodology of the SIMPLE algorithm to formulate pressure correction equations at locations 1, 2, 3 and 4. [12 marks]

b) To solve the set of equations obtained in (a), tabulate the coefficients required for the application of the TDMA method in the form of **Table Q2.1.** [2 marks]

c) Use the TDMA algorithm to solve the set of equations to obtain p'_1, p'_2, p'_3 and p'_4 [4 marks]

d) Using the solution obtained in (c), calculate the corrected velocities at B, C, D and E and show mass is conserved. [2 marks]

The TDMA method

For a set of equations of the form

$$-\beta_j\phi_{j-1} + D_j\phi_j - \alpha_j\phi_{j+1} = C_j$$

where β_j , α_j and D_j are coefficients, *j = 2,3,4 ....(n-1)* are points along a line.

ϕ_j can be obtained from the recurrence formulae:

$$\phi_j = A_j\phi_{j+1} + C'_j$$

$$A_j = \frac{\alpha_j}{\left(D_j - \beta_j A_{j-1}\right)}$$

$$C'_j = \frac{\left(\beta_j C'_{j-1} + C_j\right)}{\left(D_j - \beta_j A_{j-1}\right)}; \quad A_1 = 0 \text{ and } \quad C''_1 = \phi_1$$

3. A square cross-sectional bar is attached to a surface as shown in **Figure Q.3**. The cross-sectional dimensions of this bar are 1 cm x 1 cm square. The properties are, thermal conductivity k=100 W/(m.°C), $\rho c = 50\times10^3$ J/(m^3.°C). All axial surfaces of this bar are insulated and the end of the bar is exposed to an ambient temperature of 20 °C with a convective heat transfer coefficient h = 40 W/(m^2.°C). Initially the bar is at a uniform temperature of 20 °C and at $t = 0$ a heat flux of 4 kW/m^2 is applied at the base and maintained.

Transient heat transfer in this situation is governed by

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + q = \rho c\frac{\partial T}{\partial t}$$

Five equally spaced control volumes are to be used to formulate discretised equations and obtain a numerical solution.

a) Write the discretised form of the governing equation at a general node using the explicit method and define its coefficients. [4 marks]

b) Incorporate appropriate boundary conditions after $t > 0$ and write the numerical form of discretised equations for each node. [10 marks]

c) Determine a suitable time step to calculate the transient temperature distribution of the bar. [3 marks]

d) Using a suitable time step calculate the temperature distribution at time $t = 10$ s. [3 marks]

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Attach This to the Answer Book.

ID Number of the candidate: ____________________________

TABLE Q.2.1

Point	β_j	D_j	α_j	C_j	A_j	C'_j
1						
2						
3						
4						

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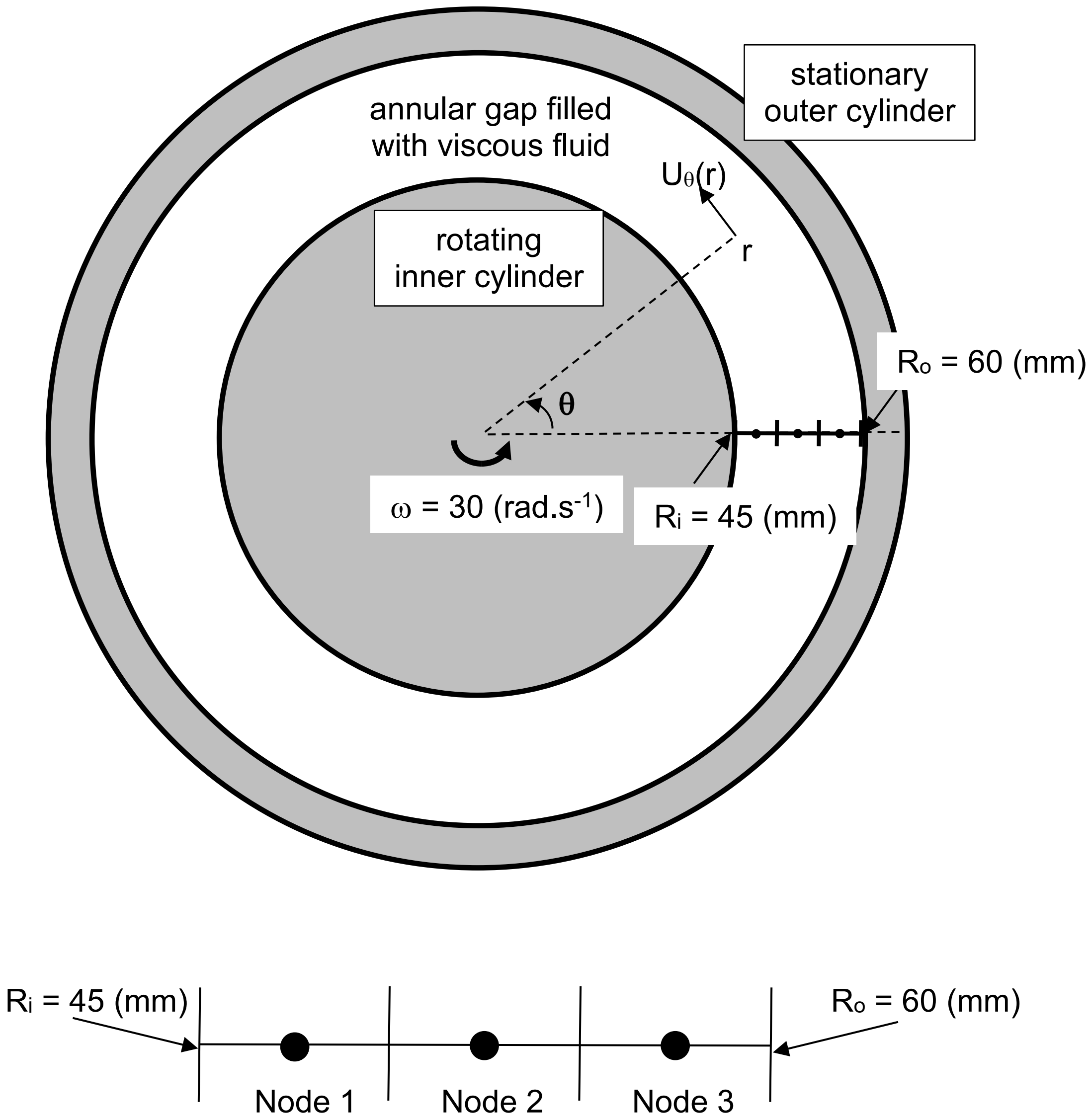

Figure Q.1

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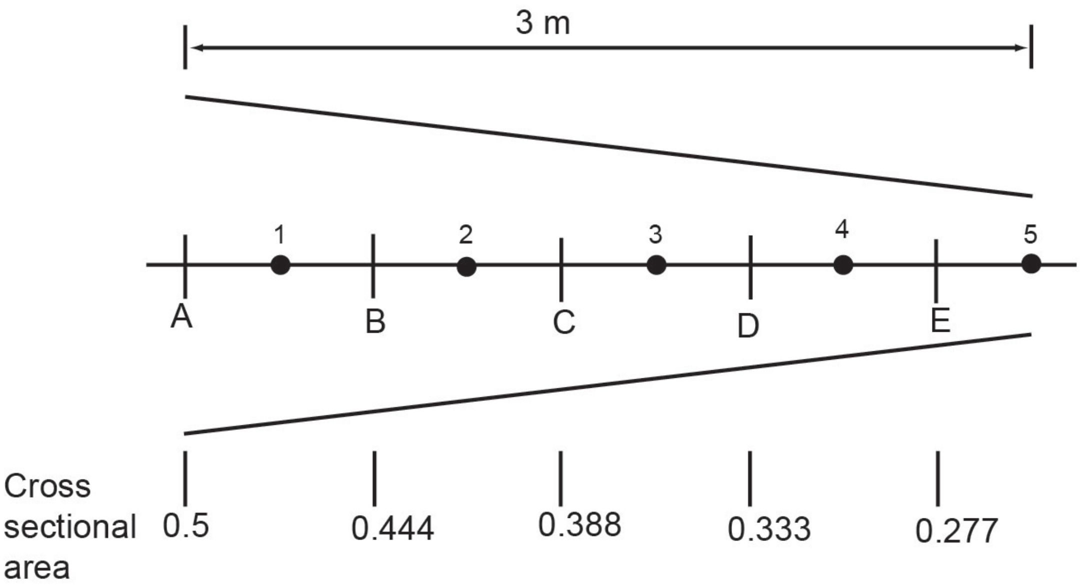

Figure Q.2

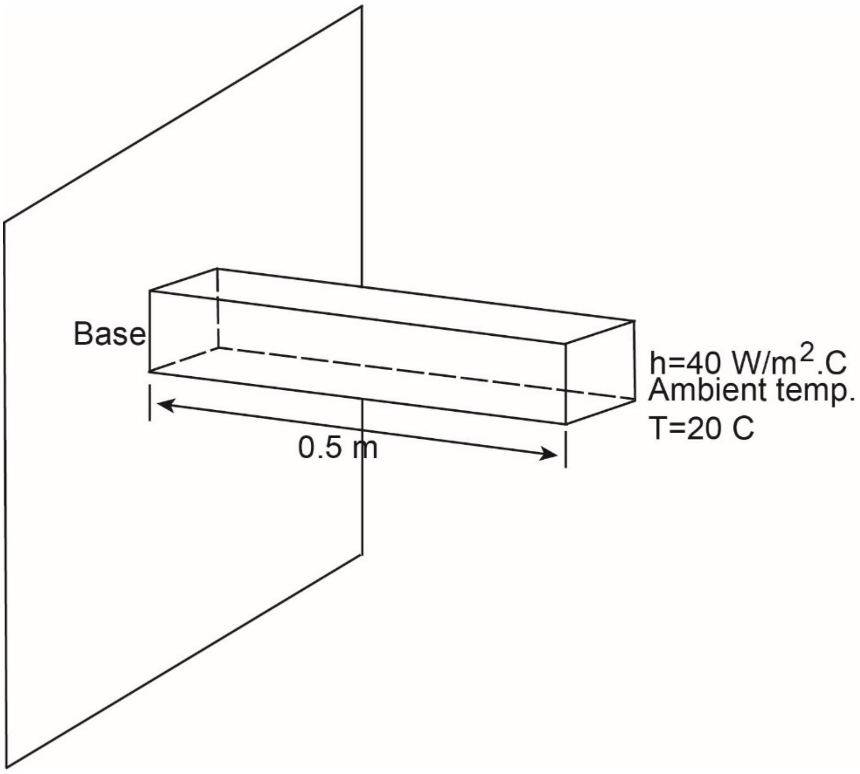

Figure Q.3

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USEFUL EQUATIONS

Note: All equations are in standard notations used during the lecture course.

Steady State One-dimensional Diffusion Equation is:

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0$$

where Γ is the diffusion coefficient, S is the source term.

Discretised form of the one-dimensional diffusion equation is:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u}$$

where

a_W	a_E	a_P
$\frac{\Gamma_w}{\delta x_{WP}}A_w$	$\frac{\Gamma_e}{\delta x_{PE}}A_e$	$a_W+a_E-S_P$

Boundary conditions can be introduced by cutting links with the appropriate face(s) and modifying the source terms. Equivalent expressions for 2-D and 3-D can be wrriten based on 1-D expressions. In conduction problems Γ is thermal conductivity k.

Steady State One-Dimensional Convection and Diffusion Equation

In the absence of sources, steady convection and diffusion of a property ϕ in a given one-dimensional flow field u is governed by:

$$\frac{d}{dx}(\rho u\phi)=\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)$$

The flow must also satisfy continuity $\frac{d(\rho u)}{dx}=0$

Discretised form of one-dimensional steady state convection diffusion equation is:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E}$$

with $$\boxed{a_P=a_W + a_E + (F_e - F_w)}$$

$$F_w=(\rho uA)_w, \qquad F_e=(\rho uA)_e$$

$$D_w=\frac{\Gamma_w}{\delta x_{WP}}A_w, \qquad D_e=\frac{\Gamma_e}{\delta x_{PE}}A_e$$

Coefficients depends on the discretisation scheme used.

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If source terms are present in the governing equation they could be accommodated using the standard practice.

The neighbour coefficients of the discretised equation for some common schemes are:

Scheme	a_W	a_E
Central differencing	$D_w + F_w/2$	$D_e - F_e/2$
Upwind differencing	$D_w + \max(F_w, 0)$	$D_e + \max(0, -F_e)$
Hybrid differencing	$\max[F_w, (D_w + F_w/2), 0]$	$\max[-F_e, (D_e - F_e/2), 0]$
Power law $Pe = \dfrac{F}{D} = \dfrac{\rho u}{\Gamma/\delta x}$	$D_w \max[0, (1 - 0.1\lvert Pe_w\rvert)^5] + \max(F_w, 0)$	$D_e \max[0, (1 - 0.1\lvert Pe_e\rvert)^5] + \max(-F_e, 0)$
Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.		

The discretised equation of the one-dimensional steady state convection diffusion equation using the standard QUICK scheme at a general internal node point is:

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + a_{WW} \phi_{WW} + a_{EE} \phi_{EE}$$

where $a_P = a_W + a_E + a_{WW} + a_{EE} + (F_e - F_w)$

The neighbour coefficients of the standard QUICK scheme in 1-D are:

	Standard QUICK
a_W	$D_w + \frac{6}{8}\alpha_w F_w + \frac{1}{8}\alpha_e F_e + \frac{3}{8}(1-\alpha_w)F_w$
a_{WW}	$-\frac{1}{8}\alpha_w F_w$
a_E	$D_e - \frac{3}{8}\alpha_e F_e - \frac{6}{8}(1-\alpha_e)F_e - \frac{1}{8}(1-\alpha_w)F_w$
a_{EE}	$\frac{1}{8}(1-\alpha_e)F_e$

with $\alpha_w = 1$ for $F_w > 0$ and $\alpha_e = 1$ for $F_e > 0$

$\alpha_w = 0$ for $F_w < 0$ and $\alpha_e = 0$ for $F_e < 0$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

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The SIMPLE algorithm

Pressure correction p' is the difference between correct pressure field p and the guessed pressure field $p*$, so that

$$\boxed{p = p* + p'}$$

Velocity corrections in two-dimensions u' and v' relate to the correct velocities u and v when the guessed velocities $u*$ and $v*$ as

$$\boxed{u = u* + u'}$$

$$\boxed{v = v* + v'}$$

Velocity corrections are obtained from pressure corrections field using

$$\boxed{u_{i,J} = u^*_{i,J} + d_{i,J}(p'_{I-1,J} - p'_{I,J})}$$

$$\boxed{v_{I,j} = v^*_{I,j} + d_{I,j}(p'_{I,J-1} - p'_{I,J})}$$

In a two-dimensional flow the continuity equation is:

$$\boxed{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0}$$

In a two-dimensional grid using west (W), east (E), south (S) and north (N) notations, the pressure correction equation derived from the continuity equation takes the form:

$$\boxed{a_P p'_P = a_W p'_W + a_E p'_E + a_S p'_S + a_N p'_N + b'}$$

where $\boxed{a_{W'} = (\rho dA)_w}$; $\boxed{a_{E'} = (\rho dA)_e}$; $\boxed{a_{S'} = (\rho dA)_s}$; $\boxed{a_{N'} = (\rho dA)_n}$

$$\boxed{a_P = a_W + a_E + a_S + a_N}$$

$$\boxed{b' = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n}$$

One-Dimensional Unsteady Heat Conduction is governed by:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + S$$

Discretised equation using the explicit scheme for one-dimensional unsteady heat conduction is

$$\boxed{a_P T_P = a_W T_W^o + a_E T_E^o + \left[a_P^o - (a_W + a_E - S_p)\right] T_P^o + S_u} \qquad (8.1)$$

where $\boxed{a_P = a_P^o}$

and $$a_P^o = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

Discretised equation using the fully implicit scheme for one-dimensional unsteady heat conduction is

$$a_P T_P = a_W T_W + a_E T_E + a_P^o T_P^o + S_u$$

where $$a_P = a_P^o + a_W + a_E - S_p$$

and $$a_P^o = \rho c \frac{\Delta x}{\Delta t}$$

with

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

Tri-Diagonal Matrix Algorithms (TDMA) for the Solution of Linear Equations

For a system of equations that has a tri-diagonal form any single equation may be written in the form:

$$-\beta_j \phi_{j-1} + D_j \phi_j - \alpha_j \phi_{j+1} = C_j$$

The solution can be obtained from the recurrence relationships:

$$\phi_j = A_j \phi_{j+1} + C'_j$$

where $$A_j = \frac{\alpha_j}{D_j - \beta_j A_{j-1}}$$

$$C'_j = \frac{\beta_j C'_{j-1} + C_j}{D_j - \beta_j A_{j-1}}$$

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At the boundary points $j = 1$ and $j = n+1$ the values for A and C' are:

$$A_1 = 0 \text{ and } C'_1 = \phi_1 \text{ and } A_{n+1} = 0 \text{ and } C'_{n+1} = \phi_{n+1}$$