MATHEMATICS AND MODELLING FOR MATERIALS
(23MAB101)

Semester 1 23/24 | In-Person Exam paper

Please fill in:

ID number: [ ] Desk number: [ ]

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

Write your answer for every question in the appropriate space. Indicate within that space if you give additional parts of your answer elsewhere.

Answer ALL QUESTIONS

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1. **[25]**

(a) Find the Taylor series of $f(x,y) = x^2 \sin y + 2$ near the point $(1, \frac{\pi}{2})$ up to second order. [5]

Continue at the end of the booklet (if necessary).

(b) For the function $f(x,y) = (x^2+y^2)^2-(x^2-y^2)$, find the coordinates of the stationary points and determine their nature (minimum, maximum or saddle point). [10]

Continue at the end of the booklet (if necessary).

(c) Show that the function

$$u(x,t) = e^{-\alpha^2 t} \sin x$$

satisfies the differential equation

$$u_t = \alpha^2 u_{xx}$$ [10]

Continue at the end of the booklet (if necessary).

2. **[25]**

(a) Sketch the graph of the periodic function

$$f(x) = \begin{cases} -\pi\,, & -\pi \leq x < 0 \\ \pi\,, & 0 \leq x < \pi \end{cases} \quad , \quad f(x+2\pi) = f(x),$$

over the interval $-2\pi < x < 2\pi$. [5]

Continue at the end of the booklet (if necessary).

(b) Obtain the Fourier series for the function $f(x)$ above. [15]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

(c) What value does the Fourier series of $f(x)$ converge to at $x = 0$ and at $x = 2$? Explain your reasoning. [5]

Continue at the end of the booklet (if necessary).

3. **[25]**

(a) Find the Fourier transform of

$$f(t) = \begin{cases} 1 - t\,, & 0 \leq t < 1 \\ 1 + t\,, & -1 \leq t < 0 \\ 0\,, & \text{otherwise} \end{cases}$$ [15]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

(b) Find the Fourier transform of

$$f(t) = \begin{cases} 2 - t\,, & 0 \leq t < 1 \\ 0\,, & \text{otherwise} \end{cases}$$ [10]

Continue at the end of the booklet (if necessary).

4. **[25]**

(a) Use the method of separation of variables to obtain the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{9}\frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le 1$ and $t > 0$, which satisfies the boundary conditions

$$u(0,t) = 0\,, \quad u(1,t) = 0\,,$$

and initial condition

$$u(x,0) = x$$ [15]

Hint: When you impose the initial condition, you should use the half-range Fourier series method.

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

(b) Use direct integration to solve the following partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x \cos y \,,$$

subject to the boundary conditions:

$$u(\pi, y) = 2\sin y \,, \quad \frac{\partial u}{\partial x}\left(x, \tfrac{\pi}{2}\right) = 2x$$ [10]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).