Loughborough University

MATHEMATICS FOR MECHANICAL ENGINEERING 3
(23MAB110)

Semester 1 23/24 — In-Person Exam paper

Please fill in:

ID number:		Desk number:	

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

Write your answer for every question in the appropriate space. Indicate within that space if you give additional parts of your answer elsewhere.

Answer ALL QUESTIONS

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1. **[25]**

(a) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} .$$

[5]

Continue at the end of the booklet (if necessary).

(b) Using your results from part (a), find the solution to the system of second-order ordinary differential equations

$$\ddot{x}(t) = 2x(t) - 3y(t)$$
$$\ddot{y}(t) = x(t) - 2y(t)$$

subject to the initial conditions

$$x(0) = 0\,, \quad y(0) = 0\,, \quad \dot{x}(0) = 1\,, \quad \dot{y}(0) = 0\,.$$ [10]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

(c) Write down a real matrix with eigenvalues -1 and 3, and corresponding eigenvectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. [5]

Continue at the end of the booklet (if necessary).

(d) Find the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ [5]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

2. **[25]**

(a) Sketch the graph and obtain the half-range Fourier cosine series for the function

$$f(x) = \sin(2x) \quad 0 \leq x < \pi.$$

[20]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

(b) Consider the periodic function

$$g(x) = \begin{cases} 1 - x^2\,, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 3\,, & -\pi \leq x < -\frac{\pi}{2}\,,\ \frac{\pi}{2} < x < \pi \end{cases}\,, \qquad g(x + 2\pi) = g(x),$$

What value does the Fourier series of $g(x)$ converge to at $x = \frac{\pi}{2}$? Explain your reasoning. [5]

Continue at the end of the booklet (if necessary).

3. **[25]**

(a) Reverse the order of integration to evaluate the integral

$$I_1 = \int_0^1 \int_{x^{2/3}}^1 x\cos(y^4)\,dydx\,.$$ [5]

Continue at the end of the booklet (if necessary).

(b) Find the moment of inertia with respect to the x axis of a plate having for edges the curve

$$y = -x(x-1)$$

and the x axis, if its density is inversely proportional to the distance from the x axis.

[15]

Continue at the end of the booklet (if necessary).

(c) Use polar coordinates to evaluate

$$\int_{-1}^{1}\int_{0}^{\sqrt{(1-x^2)}} (x^2+y^2)^3\,dydx$$ [5]

Continue at the end of the booklet (if necessary).

4. **[25]**

(a) Use the method of separation of variables to obtain the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 25\,\frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le 1$ and $t > 0$, which satisfies the boundary conditions

$$u(0,t) = 0\,, \quad u(1,t) = 0\,,$$

and initial conditions

$$u(x,0) = 7\sin(3\pi x)\,, \quad u_t(x,0) = -\frac{1}{2}\sin(2\pi x)\,,$$ [15]

Continue on the next page (if necessary).

Continue at the end of the booklet (if necessary).

(b) A continuous random variable X has probability density function

$$f(x) = \frac{A}{x^3}\,, \quad 1 \leq x \leq 4$$

where A is a constant. Find the value of A and hence determine the expectation $E(X)$ and the variance $V(X)$. Obtain the probability $P(1 < X < 2)$. Express your results either as exact numbers or round to 3 d.p. [5]

Continue at the end of the booklet (if necessary).

(c) If the probability of a defective bolt is 0.1, find the mean and standard deviation for the number of defective bolts in a total of 400 bolts. [5]

Continue at the end of the booklet (if necessary).

Extra space for answers to any questions: