Loughborough University

PROBABILITY THEORY
(23MAB170)

Semester 1 2023/2024 | In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are **not** allowed).

Answer **THREE** questions.

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1. (a) Consider four coins, three of which are fair, yielding heads with probability $1/2$, while the fourth is biased, yielding heads with probability $3/4$. In an experiment we take a random coin (each with probability $1/4$) and toss it three times.
   (i) What is the probability to observe three heads? [5]
   (ii) Given that all three tosses yielded heads, what is the probability that we have chosen the biased coin? [5]
   (iii) Given that *not all* three tosses yielded heads, what is the probability that we have chosen the biased coin? [5]

   (b) Prove *Boole's inequality:*

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) \leq \mathbb{P}(A_1) + \cdots + \mathbb{P}(A_n). \qquad [5]$$

2. (a) A drunkard starts at the origin $(0, 0)$ and makes two independent random steps. The steps are of length 1, and the direction is chosen between north, west, south and east uniformly. For example, after the first step the drunkard is at one of the points $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ with probability $1/4$ each. Let X, Y be the coordinates after the two steps and $R = \sqrt{X^2 + Y^2}$ be the distance from the origin.

   (i) Find the joint probability mass function $p_{X,Y}$ of X, Y. [4]

   (ii) Find $\mathbb{E}(R)$. [4]

   (b) Suppose that X and Y are independent random variables with distribution functions F_X and F_Y. Find distribution functions of $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. [4]

   (c) A dart is thrown into the square $S = \{(x, y) : |x| + |y| \le 1\}$, the point (X, Y) where the dart lands is distributed uniformly within S.

   (i) Find the joint density $f_{X,Y}$. Make sure to give an expression for $f_{X,Y}(x, y)$ which is valid for all $(x, y) \in \mathbb{R}^2$. [4]

   (ii) Are X and Y independent? (Justify!) [4]

3. (a) Let X, Y be random variables with means μ_X and μ_Y, variances σ_X^2 and σ_Y^2 and covariance $\mathrm{Cov}(X, Y) = c$. Find the best mean square approximation of X by a linear function of Y, i.e. find $a, b \in \mathbb{R}$ which minimize $\mathbb{E}\big((aY + b - X)^2\big)$:

$$\mathbb{E}\big((aY + b - X)^2\big) = \min_{s,t} \mathbb{E}\big((sY + t - X)^2\big). \quad [5]$$

   (b) Give an example of two random variables X and Y with $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathbb{E}(X^2) = \mathbb{E}(Y^2)$, but such that $\mathbb{P}(X \le t) \neq \mathbb{P}(Y \le t)$ for some t. [5]

   (c) Let X be a random variable with a moment generating function $\mathbb{E}(e^{tX}) = e^{e^t - c}$.

   (i) Find c. [5]

   (ii) Find $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$. [5]

4. (a) Let $X_1, X_2, \ldots$ be independent identically distributed random variables, uniformly distributed in $[-1, 2]$. Prove that for each $t \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{P}(X_1 + \cdots + X_n \le t) = 0. \quad [5]$$

   (b) Let $X_1, X_2, \ldots$ be independent dice rolls (uniform on $\{1, 2, 3, 4, 5, 6\}$). Let $M_n = \max\{X_1, \ldots, X_n\}$ for $n \ge 1$, and $D_1 = X_1$, $D_n = X_n - X_{n-1}$ for $n > 1$.

   (i) Is $M_1, M_2, \ldots$ a Markov chain? If yes, write down the matrix of transition probabilities. [5]

   (ii) Is $D_1, D_2, \ldots$ a Markov chain? If yes, write down the matrix of transition probabilities. [5]

   (iii) Compute $\mathbb{P}(M_4 = 6 \mid M_2 = 2)$. [5]