Loughborough University

ANALYTICAL DYNAMICS
(23MAB255)

Semester 2 23/24 | In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

Answer **3** questions.

You may assume any formula in the yellow formula book.

1. (a) A thin uniform piece of wire of length l and mass M is bent into the shape of a semicircle.

   (i) Find the position of the centre of mass of this wire. [5]

   (ii) Find the moment of inertia of this wire about its axis of symmetry. [5]

   (iii) Find the moment of inertia of this wire about the axis passing through its end-point and parallel to its axis of symmetry. [3]

   (b) Show that the centre of mass of a uniform solid right circular cone of height h is at a distance $h/4$ from the base. [7]

2. A circle can rotate freely around the vertical axis that coincides with a diameter of the circle. A bead slides freely along this circle. The circle does not move up or down. The radius of the circle is a. The moment of inertia of the circle about its diameter is I. The mass of the bead is m. The acceleration due to gravity is g. All constraints are ideal. No external forces other than gravity act on the system. The setup is illustrated in Fig. Q2.

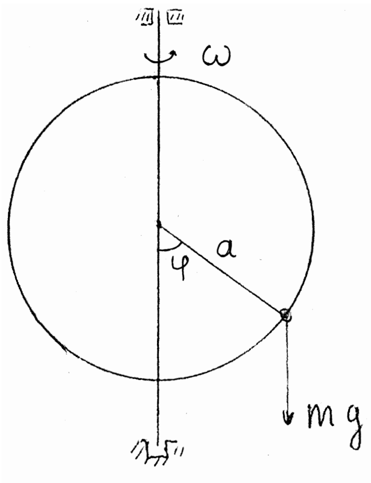

Fig. Q2

(a) Explain why the angular momentum of the system about the axis of rotation and the total energy of the system are conserved during the motion. [3]

(b) At an initial moment of time $t = 0$ the circle has angular velocity ω_0. At this moment of time the bead slides down along the circle with speed u_0 with respect to the circle from the position when the angle φ in Fig. Q2 is equal to φ_0.

(i) Find the values at time $t = 0$ of the total angular momentum of the system about the axis of rotation and the total kinetic energy of the system. [5]

(ii) The angular velocity ω of rotation of the circle and the speed u of the bead with respect to the circle change as the angle φ changes in the process of motion. Use the conserved quantities from part 2(a) to express ω and u as functions of φ. [8]

(iii) Assume that $\varphi_0 = \pi/2$. Prove that if $u_0 < \sqrt{2ga}$ then in the subsequent motion the bead cannot make it the whole way round the circle. [4]

3. A rectilinear tube can rotate freely in a horizontal plane around the vertical axis through an endpoint O of the tube. The tube does not move up or down. A bead slides inside the tube. It is connected to the point O by a light spring. The moment of inertia of the tube about the axis of rotation is I. The mass of the bead is m. The natural length of the spring is l. The spring satisfies Hooke's law with spring constant k. All constraints are ideal. No external forces other than gravity and the restoring force of the spring act on the system. The setup is illustrated in Fig. Q3.

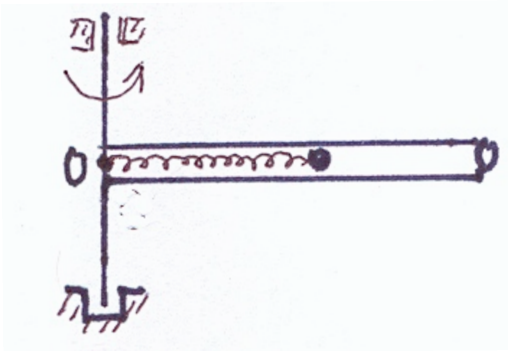

Fig. Q3

(a) How many degrees of freedom does this system have? Describe a possible system of generalised coordinates for this system. [3]

(b) Find the kinetic energy and potential energy of the system. [6]

(c) Find the Lagrangian for the system. Write down Lagrange's equations of motion. [5]

(d) Find the Hamiltonian for the system. Write down Hamilton's equations of motion. [6]

4. (a) Consider a natural Lagrange's system with potential energy V and equilibrium position q_*. Let the quadratic form $V^{(2)}$ of V at q_* be non-degenerate. Formulate Lagrange's theorem about stability of such an equilibrium. [3]

(b) A system with two degrees of freedom has kinetic energy T and potential energy V:

$$\begin{aligned} T &= \frac{1}{2}ml^2\left[(1+\sin^2 q_1)\dot{q}_1^2 + (1+\sin^2 q_2)\dot{q}_2^2\right], \\ V &= -mgl(\cos q_1 + \cos q_2) + \frac{1}{2}kl^2(q_1 - q_2)^2 \end{aligned}$$

with generalised coordinates q_1, q_2 and positive constants m, g, l, k.

(i) Show that $q_1 = 0, q_2 = 0$ is an equilibrium position of the system. Is this equilibrium stable? Justify your answer. [5]

(ii) Write down the linearised Lagrange equations for the system in a small neighbourhood of this equilibrium position. [6]

(iii) Find the eigenfrequencies of small oscillations near this equilibrium position. [6]