Loughborough University

INTRODUCTION TO DYNAMICAL SYSTEMS
(23MAC148)

Semester 1 23/24 In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

Answer **3** questions.

You may assume any formula in the yellow formula book.

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1. (a) Consider a map $f : I \to \mathbb{R},\ I \subseteq \mathbb{R}$. Explain whether or not this map determines a dynamical system according to the formula $x_{n+1} = f(x_n),\ x \in I$ for the following cases:

   (i) $f(x) = |x - 0.1|,\ I = [-1, 1]$; [2]

   (ii) $f(x) = x^2 - 2.5x + 2,\ I = [0, 2]$. [3]

   (b) Find all fixed points of the dynamical system $x_{n+1} = 1 + (x_n - 1)\,(2 - e^{x_n - 4})$, $x \in \mathbb{R}$ and investigate their stability. [4]

   (c) Draw cobweb plots and use these plots to establish whether the fixed point $x = 0$ is an attractor, or a repeller, or neither of them for the following dynamical systems:

   (i) $x_{n+1} = x_n + x_n^2,\ x \in \mathbb{R}$; [3]

   (ii) $x_{n+1} = x_n + x_n^3,\ x \in \mathbb{R}$. [2]

   (d) (i) Consider a dynamical system $x_{n+1} = f(x_n),\ x \in \mathbb{R}$. Let $\{\overline{x_0, x_1, \ldots, x_{p-1}}\}$ be a periodic orbit. Assume that the function f is differentiable at all points $x_0, x_1, \ldots, x_{p-1}$. Give a brief derivation of the formula $\mu = f'(x_0)f'(x_1)\ldots f'(x_{p-1})$ for the multiplier of this periodic orbit. [3]

   (ii) Consider the dynamical system $x_{n+1} = \pi(b\sin(x_n/2) - \cos x_n)$, $x \in \mathbb{R}$, where b is a constant. Find values of b such that $\{\overline{0, -\pi}\}$ is a period-2 orbit of this dynamical system. Explain whether this periodic orbit is an attractor or a repeller. [3]

2. The base-5 shift map $f : [0, 1) \to [0, 1)$ is defined by the formula

$$f(x) = \mathrm{Frac}(5x).$$

Here $\mathrm{Frac}(u)$ denotes the fractional part of u.

   (a) Define the sequence space Σ_5 and the map $\sigma : \Sigma_5 \to \Sigma_5$ such that the action of σ on sequences in Σ_5 coincides with the action of f on sequences of digits in the base-5 representation of numbers in the interval $[0, 1)$. [3]

   (b) (i) Using the symbolic dynamics in Σ_5 determine all the fixed points for σ and hence for f, expressing your answers both in symbolic form and as fractions. [3]

   (ii) Find the maximal $x \in [0, 1)$ such that x is a period-3 point of f. [4]

   (c) Let $\mathbf{s} = (s_1 s_2 \ldots)$ and $\mathbf{t} = (t_1 t_2 \ldots)$ be two points of Σ_5. The distance in Σ_5 between $\mathbf{s}$ and $\mathbf{t}$ can be defined as $d[\mathbf{s}, \mathbf{t}] = \sum\limits_{i=1}^{\infty} \frac{|s_i - t_i|}{5^i}$.

   (i) Prove that if $s_i = t_i$ for $1 \le i \le n$, then $d[\mathbf{s}, \mathbf{t}] \le 1/5^n$. [3]

   (ii) Prove that if $d[\mathbf{s}, \mathbf{t}] < 1/5^n$, then $s_i = t_i$ for $1 \le i \le n$. [2]

   (d) Let $\mathbf{s}$ be a point of Σ_5. Prove that for any $\varepsilon > 0$ there exist a point $\mathbf{t}$ of Σ_5 and a natural number n such that $d[\mathbf{s}, \mathbf{t}] < \varepsilon$ and, for any $k \ge n$, if k is even then $d[\sigma^k(\mathbf{s}), \sigma^k(\mathbf{t})] = 1/12$, or if k is odd then $d[\sigma^k(\mathbf{s}), \sigma^k(\mathbf{t})] = 5/12$. [5]

3. (a) Consider the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \ln(2-y^2)+x \\ y+(3^x-3^y)/\ln 3 \end{pmatrix}.$$

   (i) Find all the fixed points of this map. [4]

   (ii) Study stability of these fixed points. [7]

(b) Find the area in $\mathbb{R}^2$ of the image of the set $\{(x,y) : |x|+2|y| \leq 1\}$ under the transformation by shift along solutions of the system

$$\begin{aligned} \dot{x} &= x\sin^3 t - y\cos^3 t\,, \\ \dot{y} &= x\sin^3 t + y\cos^3 t \end{aligned}$$

from time $t=0$ until time $t=\pi/2$. [7]

(c) Let the matrix of the monodromy map of a linear homogeneous time-periodic ODE have the form

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \qquad \alpha \in (0,\pi).$$

Is the equilibrium of this ODE at the origin of coordinates stable? Justify your answer. [2]

4. Draw bifurcation diagrams for the following systems depending on the parameter α. Indicate a standard name for each of the observed bifurcations.

(a) $\dot{x} = \alpha + 3 + \cosh^2 x$ [4]

(b) $\dot{x} = (x-\alpha)(x^2-2x-3\alpha+6)$ [9]

(c) $\dot{x}_1 = x_1^2 - (5+\alpha)x_1 + 7 + 2\alpha,\ \dot{x}_2 = 1 - 2x_2$ [7]