ADVANCED COMPLEX ANALYSIS
(23MAC241)

Semester 2 23/24 — Online Long-window Exam paper

This is an online long-window examination, meaning you have **23 hours** in which to complete and submit this paper. How you manage your time within the 23-hour window is up to you but we expect you should only need to spend approximately **2 hours** working on it. If you have extra time or rest breaks as part of a Reasonable Adjustment, you will need to add this to the amount of time you are expected to spend on the paper

It is your responsibility to submit your work by the deadline for this examination. You must make sure you leave yourself enough time to do so

It is also your responsibility to check you have submitted the correct file.

Exam help

If you are experiencing difficulties in accessing or uploading files during the exam period you should contact the Exam Helpline. For urgent queries please call **01509 222 900**.

For other queries e-mail `examhelp@lboro.ac.uk`

You may handwrite your answers.

You may use a calculator for this exam.

Answer **ALL FOUR** questions.

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1. (a) Let

$$u(x, y) = x - x^2 + y^2.$$

(i) Find a function $v(x, y)$ such that $u(x, y) + \mathrm{i}v(x, y) = f(x + \mathrm{i}y)$ for some regular function $f(z)$. [10]

(ii) Write down $f(z)$ explicitly and state where it is regular. [4]

(b) Show that the function $f(z) = \exp(\bar{z})$ is not regular anywhere in $\mathbb{C}$. [5]

(c) Let $g(z)$ be an entire function that satisfies the inequality $|g(z)| \geq c > 0$ for all $z \in \mathbb{C}$ for some positive constant c. Use Liouville's theorem to prove that g is a constant function. [6]

2. Define $\lg : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by

$$\lg(z) = \ln|z| + \mathrm{i}\arg^*\{z\},$$

where $\arg^*\{z\}$ is the unique argument of z in the interval $[0, 2\pi)$.

(a) Prove that $\lg(z)$ defines a branch of logarithm and that it is regular on the domain

$$\mathbb{C} \setminus \{z : \mathfrak{Im}\{z\} = 0, \mathfrak{Re}\{z\} \geq 0\}.$$ [6]

(b) Using an appropriately-chosen contour integral, show that

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)(1+x^2)}\,\mathrm{d}x = \frac{\pi}{2}(\sqrt{2}-1).$$ [19]

3. (a) Use the Residue Theorem to find the value of the infinite series

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+\frac{1}{2})^5}.$$

You may use a boundedness result from the lectures provided that you state it clearly and expain how it is used. [20]

(b) Use your answer to part (a) to show that

$$1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536}.$$ [5]

4. (a) (i) If $h(z)$ is continuous on a convex domain $D \ni 1$, show that the solution $g(z)$ to the ordinary differential equation

$$\frac{\mathrm{d}g}{\mathrm{d}z} = g(z) + h(z) \qquad (\heartsuit)$$

is

$$g(z) = \mathrm{e}^{z-1}\,g(1) + \int_1^z \mathrm{e}^{z-w}\,h(w)\,\mathrm{d}w, \qquad z \in D.$$ [4]

(ii) If, in addition, $h(z)$ is regular in D show that the solution to $(\heartsuit)$ is regular in D. [5]

(b) The *Elliptic Integral* $E_1(z)$ is a special function defined by

$$E_1(z) = \mathrm{e}^{-z}\int_0^\infty \frac{\mathrm{e}^{-t}}{t+z}\,\mathrm{d}t, \qquad \mathfrak{Re}\{z\} > 0.$$

(i) Use the Weierstraß M-test to show that $E_1(z)$ is regular for $\mathfrak{Re}\{z\} > 0$. [7]

(ii) Let $g(z) = \mathrm{e}^z(E_1(z) + \mathrm{Log}(z))$ and show that

$$\frac{\mathrm{d}g}{\mathrm{d}z} = g(z) + \frac{\mathrm{e}^z - 1}{z}.$$ [4]

(iii) Deduce that $g(z)$ may be extended to an entire function. What is the nature of the singularity of $E_1(z)$ at $z = 0$? [5]