DIGITAL CONTROL
23WSC356

Semester 2 | In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

<u>Help during the exam</u>

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

Answer **ALL FOUR** questions.

Questions carry the marks shown.

Use of a calculator is permitted - It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

A range of formulae and tables likely to be of benefit in the solution of these questions is provided at the rear of the paper.

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1. Short question

a) Draw a block diagram of the digital filter that represents the difference equation

$$y_k = a_0 u_k - a_1 u_{k-1} + b_1 y_{k-1} + b_2 y_{k-2}$$

where y_{k-n} is the sampled output and u_{k-n} is the sampled input (at the nth sample interval). Be sure to label the blocks and signal paths clearly. [4 marks]

b) Considering the following digital transfer function:

$$\frac{y(z)}{u(z)} = \frac{20}{1 - 4z^{-1}}$$

i. Give the z-transform expression for the step response. [3 marks]

ii. Calculate the time response of the step response for the first four samples. [4 marks]

c) Considering a pair of complex poles at $z = 0.8 \pm 0.6j$ in the z-plane:

i. Sketch a diagram of the complex z-plane showing the position of this pair of poles including the unit circle [2 marks]

ii. Comment on whether this pole pair is stable or unstable or marginally stable. [1 mark]

d) Considering the following differential equation and the corresponding s-domain transfer function:

$$\dot{y}(t) + 5y(t) = 20x(t)$$

$$G(s) = \frac{y(s)}{x(s)} = \frac{20}{s + 5}$$

i. Obtain the difference equation from the differential equation assuming trapezoidal integration. Assume a sample time of $T = 0.2$ s and calculate the coefficients to a maximum of 3 decimal places. [4 marks]

ii. Write down the transfer function in discrete time (z-domain) based on the difference equation derived in (i). [3 marks]

iii. Apply the bilinear transform to obtain a transfer function in the z-domain from the s-domain transfer function. Assume a sample time of $T = 0.2$ s and calculate the coefficients to a maximum of 3 decimal places. [4 marks]

2. This question concerns the velocity of a mass of $m = 4$ kg, as shown in Figure Q2. The speed of the mass, v (km/h), is controlled by an applied traction force, F (kN), with a friction force of $f = bv$, where $b = 2$ N/ms^{-1} is the friction coefficient

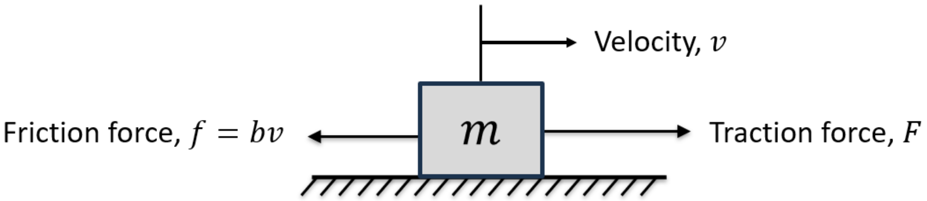

Figure Q2

We are interested in the speed v, and the traction force input, F, which are related according to the differential equation:

$$\dot{v} + \frac{b}{m}v = \frac{F}{m}$$

a) Obtain a difference equation relating the speed, v, to the traction force, F, assuming trapezoidal integration. Assume the sample time $T = 0.5$ s and calculate the coefficients to a maximum of 3 decimal places. [5 marks]

b) For the sample time $T = 0.2$ s, the difference equation of the system is

$$v_k = 0.024F_k + 0.024F_{k-1} + 0.905v_{k-1}$$

Draw a block diagram of the digital filter that represents this difference equation [4 marks]

c) Given that a suitable transfer function in the z-domain with the sample time $T = 0.2$ s is:

$$G(z) = \frac{0.024z + 0.024}{z - 0.905}$$

i. Express this transfer function in terms of the backward shift operators (z^{-1}). [3 marks]

ii. Obtain the z-transform expression for the unit pulse response and calculate the first 4 samples of the time response to a maximum of 3 decimal places. [7 marks]

d) Given the following continuous time (s-domain) transfer function of the system:

$$\frac{v(s)}{F(s)} = \frac{0.25}{s + 0.5}$$

Which has a pole at $s = -0.5$. Convert the pole to its exact equivalent in discrete time (z-domain) and mark it on a diagram of the complex z-plane. Show the unit circle on your diagram. [6 marks]

3. A robotics engineer has designed a PD controller for a servomotor with a first-order low pass filter, which has the following transfer function:

$$C(s) = \frac{6.78s + 2.37}{s + 3.49}$$

The specifications for the servomotor are: to achieve a rise time of less than 10 seconds, overshoot of less than 10%, and a steady state error of less than 2% (for a unit step), to have a phase margin of at least 45 degrees, and a gain margin of at least 6dB.

To test the design, the engineer produces a closed-loop step response for a 10-degree step change in its reference signal and a Bode plot, shown in Figures Q3(a) and Q3(b), comparing these with the response for $C(s) = 1$ i.e. unity feedback.

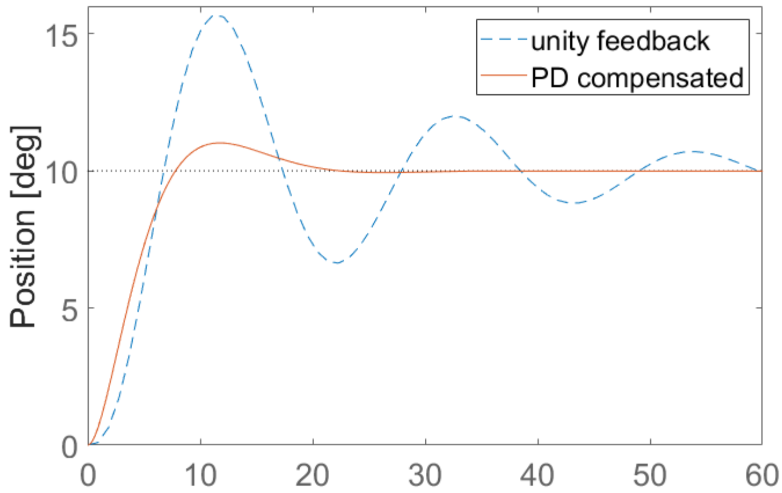

Figure Q3(a)

a) Evaluate whether the unity feedback system and the PD controlled system meet the specifications, stating any specifications that are not met in each case. [5 marks]

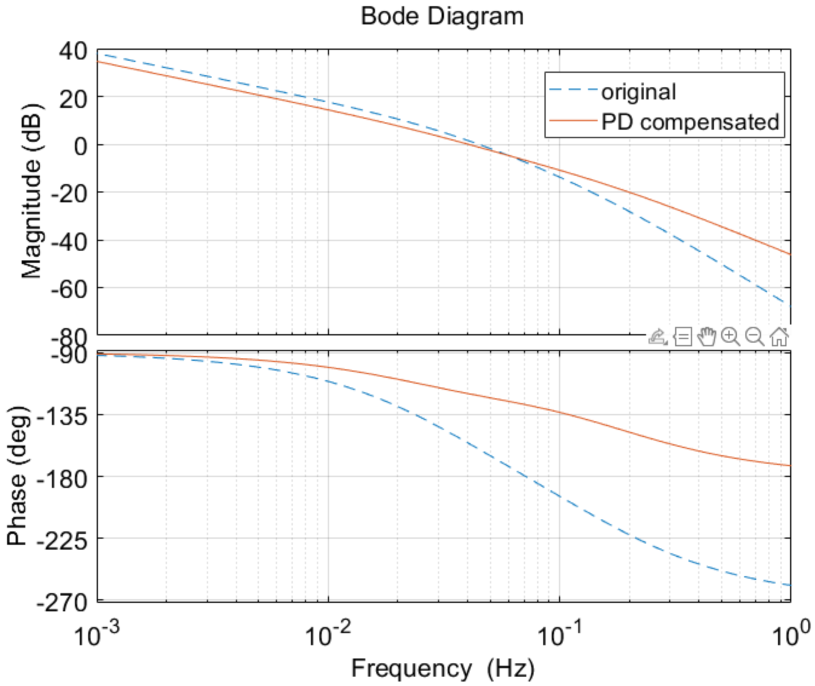

Figure Q3(b)

The designed PD compensator will be implemented on a digital microcontroller, so a discrete-time version is required

b) Choose an appropriate sampling time T_s and justify your choice. [4 marks]

c) Apply the trapezoidal approximation to show that a suitable z-domain transfer function to implement the controller is: [6 marks]

$$C(z) = \frac{(13.6 + 2.37T_s)z - 11.2}{(2 + 3.49T_s)z + 3.49\,T_s - 2}$$

d) Derive a difference equation in a form that could be used to implement this controller when programming the microcontroller. [5 marks]

e) Due to a bug in the microcontroller firmware, the sampling rate is reduced to $f_s = 0.2Hz$ when the controller is implemented. Explain the likely effect of this, considering each specification. [5 marks]

4.

An automotive engineer is in the process of designing a controller for a cruise control system. At motorway speeds, an approximate transfer function of the vehicle considering throttle valve position $u(t)$ as the control input and vehicle speed $v(t)$ as the output is:

$$G(s) = \frac{V(s)}{U(s)} = \frac{7.5}{s + 0.04}$$

The engineer considers implementing the cruise control system as a proportional control $C(z) = K$ at a sampling frequency of $f_s = 10\ \mathrm{Hz}$. This proportional controller acts on the error e between the measured $v(t)$ and a reference value $v_{ref}(t)$ as shown in Figure Q4.

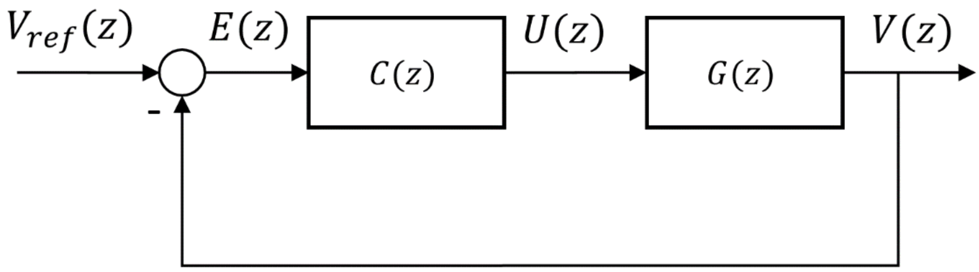

Figure Q4

a) Show that when using a Zero-Order Hold (ZOH) with a sample time T_s, the vehicle behaves according to the equivalent z-domain transfer function: [7 marks]

$$G(z) = \frac{7.5}{0.04}\left(\frac{1 - e^{-0.04T_s}}{z - e^{-0.04T_s}}\right)$$

b) Show that with a sampling frequency of $f_s = 10\ \mathrm{Hz}$, the system now behaves according to the approximate transfer function: [2 marks]

$$G(z) = \frac{0.749}{z - 0.996}$$

c) Show that the closed-loop transfer function from the reference input to the output is: [3 marks]

$$\frac{V(z)}{V_{ref}(z)} = \frac{0.749K}{z - 0.996 + 0.749K}$$

d) For what values of the proportional gain K is the system stable? [4 marks]

e) Calculate the minimum value of K required to achieve a steady-state error of less than 5% to a unit step in $V_{ref}(z)$. [4 marks]

After further testing, the engineer finds that this cruise controller works effectively if the vehicle is on a level surface but shows significant steady-state error on hills and slopes. The engineer considers correcting this problem by increasing the proportional gain K, while a colleague suggests implementing a PI controller instead.

f) Which approach would you choose: a PI controller or a proportional controller with an increased gain?

For each one, state if it will reduce the steady-state error on slopes as intended, and any possible disadvantages. [5 marks]

Useful information (control systems)

Selected Laplace and z-transforms

t-domain (t>0)	***s-domain***	***z-domain*** *(with sample time T)*
Unit step	$\frac{1}{s}$	$\frac{z}{z-1}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos(\omega T))}{z^2-2z\cos(\omega T)+1}$
$e^{-at}\ \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{z\,e^{-aT}\sin(\omega T)}{z^2-2z\,e^{-aT}\cos(\omega T)+e^{-2aT}}$
$e^{-at}\ \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z(z-e^{-aT}\cos(\omega T))}{z^2-2z\,e^{-aT}\cos(\omega T)+e^{-2aT}}$

Some results about Laplace and z-transforms

$f(t)$	$F(s)$	$F(z)$
$f(t-T)$	$e^{-Ts}\,F(s)$	$z^{-1}\,F(z)$
$e^{-at}\,f(t)$	$F(s+a)$	$F(z/e^{-a})$
$f'(t)$	$sF(s)-f(0)$	----
$\lim_{t\to\infty} f(t)$	$\lim_{s\to 0}\ \{s\,F(s)\}$	$\lim_{z\to 1}\ \{(z-1)\,F(z)\}$
Trapezoidal/Tustin/Bilinear approximation	s	$\frac{2}{T}\left(\frac{z-1}{z+1}\right)$
Pulse transfer function with Zero Order Hold	$G(s)$	$(1-z^{-1})Z\left\{\frac{G(s)}{s}\right\}_{t=kT}$

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Step response of second-order systems *- For systems well approximated by* $\frac{A\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$

Percentage overshoot:

$$O_{\%} = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Peak time:

$$T_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}}$$

Settling time to p% of steady-state:

$$T_s = \frac{\ln(100/p)}{\zeta\omega_0}$$

Rise time:

$$T_r \approx \frac{2}{\omega_0}$$

With phase margin ϕ_{PM} in degrees, $\zeta \approx \phi_{PM}/100$ for $\phi_{PM} < 60^o$, and $\phi_{PM} = 65^o$ gives $\zeta \approx 0.7$

Common SISO compensator types (PID and related controllers)

PID controller:

$$C_{pid}(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right)$$

PI controller:

$$C_{pi}(s) = K\left(1 + \frac{1}{T_i s}\right)$$

PD controller:

$$C_{pd}(s) = K(1 + T_d s)$$

Frequency-domain design formulae for PID-type controllers

(where G_C, ϕ_C are required controller gain and phase at a chosen design frequency ω_d)

PI controller:

$$T_i = \frac{1}{\omega_d \tan(-\phi_C)}$$

PID controller:

$$\tan\phi_C = T_d\omega_d - \frac{1}{T_i\omega_d}$$

PD controller:

$$T_d = \frac{\tan\phi_C}{\omega_d}$$

Controller gain (all cases):

$$K = \frac{G_c}{\sqrt{1+\tan^2\phi_C}}$$

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Additional Fomulae

Bilinear transform: $s \Rightarrow \frac{2}{T} \cdot \frac{z-1}{z+1}$ $z \Rightarrow \frac{1+sT/2}{1-sT/2}$

Precise link: $z \Leftrightarrow e^{sT}$ $s \Leftrightarrow \frac{1}{T}[\ln z]$

Trapezoidal integration: $y_k = y_{k-1} + \frac{T}{2}(u_k + u_{k-1})$

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