Loughborough University

COMPUTATIONAL FLUID DYNAMICS 1

23WSC802

Semester 2 | In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

Answer **ALL THREE** questions.

All questions carry equal marks.

You may take **TWO A4 sides of your own notes** into the examination venue.

Use of a calculator is permitted - It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

A set of useful equations is attached at the end of this paper.

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1. Conductive heat transfer in problems with spherical symmetry, constant thermal conductivity k and constant internal heat generation G is governed by the following one-dimensional equation:

$$\frac{k}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) + G = 0$$

Consider a sphere with a radius of $R_B = 6$ mm with thermal conductivity $k = 300$ W.m^{-1}.K^{-1} and rate of internal heat generation per unit volume

$G =$ 10^{10} W.m^{-3}. The boundary temperature T_B at radius R_B is kept at a constant value of 300 K.

a) Use the finite volume method on a uniform grid as sketched in **Figure Q.1** to estimate the temperatures at nodes 1, 2 and 3. (Hint: multiply the governing equation by r^2/k before carrying out the integration over a one-dimensional control volume between limits r_w and r_e). [16 marks]

b) Estimate the heat flux escaping from the sphere at radius $r = R_B$. [4 marks]

2. Shown in **Figure Q.2** is a two-dimensional cross-section of a long beam. Initially, the entire beam is at a temperature of 200 °C. Suddenly ($t > 0$), heat is generated in this beam at a rate of 20 kW/m^3, the left bottom surface of this beam is brought to a temperature of 400 °C while the right bottom surface is lowered to a temperature of 100°C. At the same time, the top surface is exposed to convective heat transfer with a convective heat transfer coefficient of $h = 200$ W/(m^2.°C), and the ambient temperature is $T_\infty = 10$°C. The properties are: thermal conductivity $k = 200$ W/(m.°C) and $\rho c = 500 \times 10^3$ J/(m^3.°C).

Two-dimensional transient heat transfer in this situation is governed by:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + g = \rho c\frac{\partial T}{\partial t}$$

A simple two-dimensional grid stencil representing the central plane of the beam as shown in **Figure Q.2** may be used to formulate discretised equations to calculate transient temperature distribution in this situation. The grid spacing $\Delta x = \Delta y = 0.1$ m.

a) Write the discretised form of the governing equation for a general node using the explicit method and provide expressions for its coefficients. [2 marks]

b) Using the grid as shown in the figure, incorporate appropriate boundary conditions after $t > 0$ and write discretised equations at each node using the explicit method. [12 marks]

c) Based on the equations obtained in (b) determine a suitable time step to calculate the transient temperature in this beam while the heat generation is maintained at the same initial rate. [2 marks]

d) Using the equations obtained in (b) and a suitable time step, calculate the temperature distribution at time $t = 10$ s. [4 marks]

3. Shown in **Figure Q.3** is part of a grid used for a two-dimensional CFD flow calculation. The velocities are calculated at staggered locations as shown. As part of this problem, a transport equation for a scalar variable ϕ is solved. Its transport is governed by

$$\frac{\partial(\rho u\phi)}{\partial x}+\frac{\partial(\rho v\phi)}{\partial y}=\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right)$$

a) Write the discretised form of the above equation for a scalar cell.

[2 marks]

The velocity values u and v shown in **Figure Q.3** are those obtained at an intermediate stage of the iterative process. Consider a scalar cell at point P and calculate the coefficients of the discretisation equation for ϕ using:

b) The central differencing scheme [3 marks]

c) the upwind differencing scheme [3 marks]

d) the hybrid differencing scheme [4 marks]

e) the QUICK differencing scheme [4 marks]

f) by making references to the coefficients calculated above, briefly comment on the suitability of each scheme. [4 marks]

Data are: Density $\rho = 1000$ kg/m^3, $\Gamma = 0.5$ kg.m^{-1}.s^{-1}

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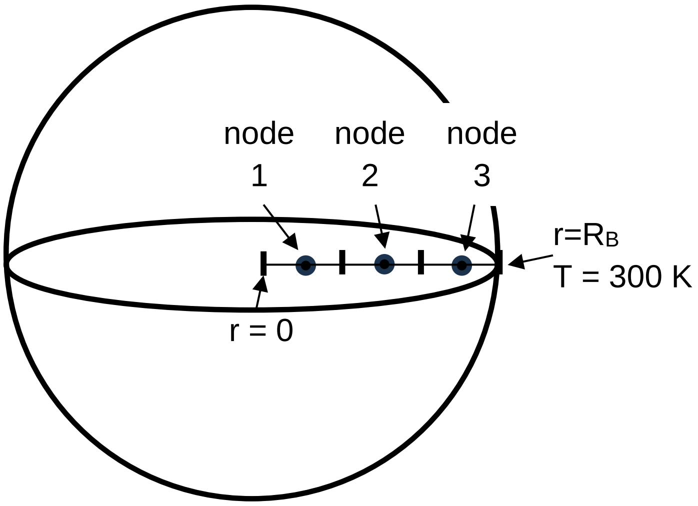

Figure Q.1

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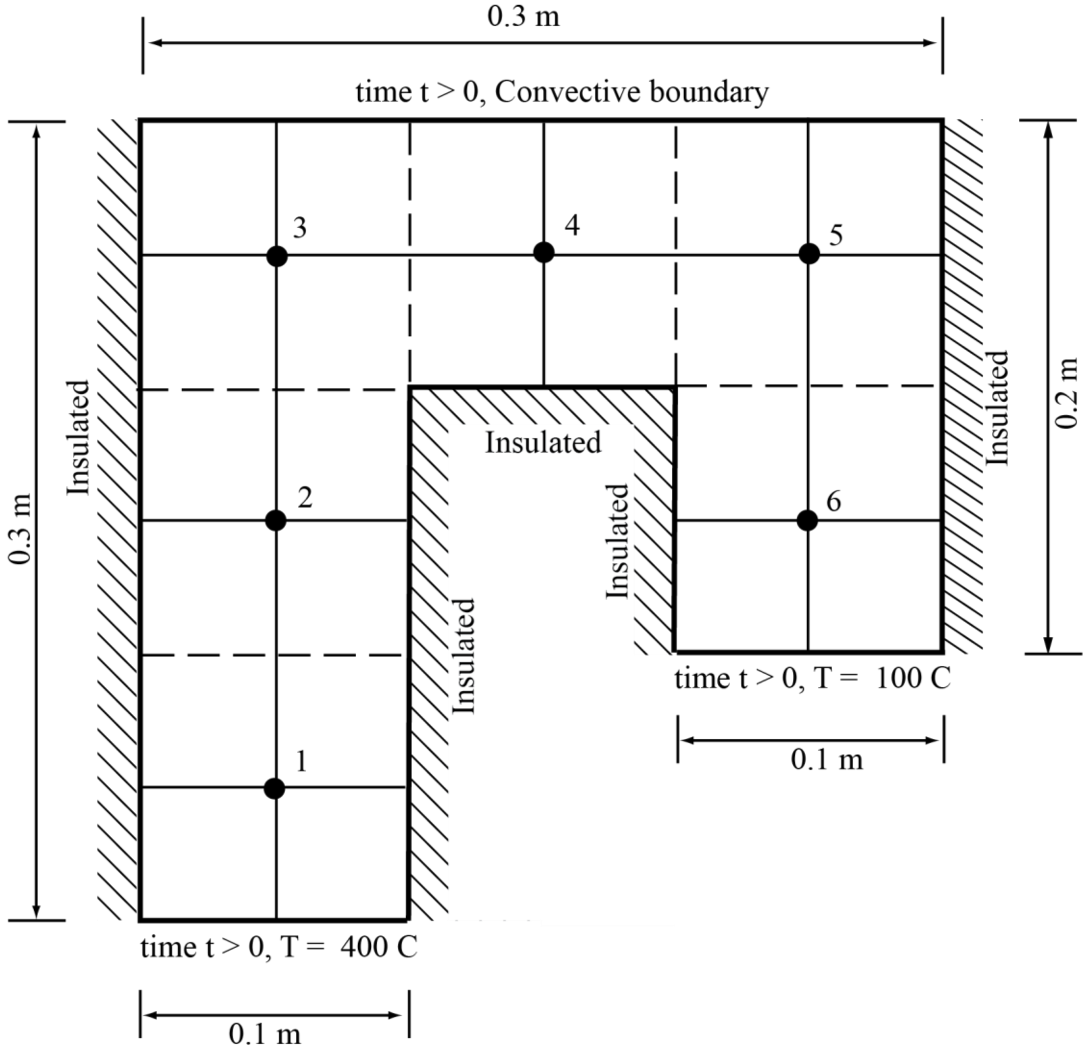

Figure Q.2

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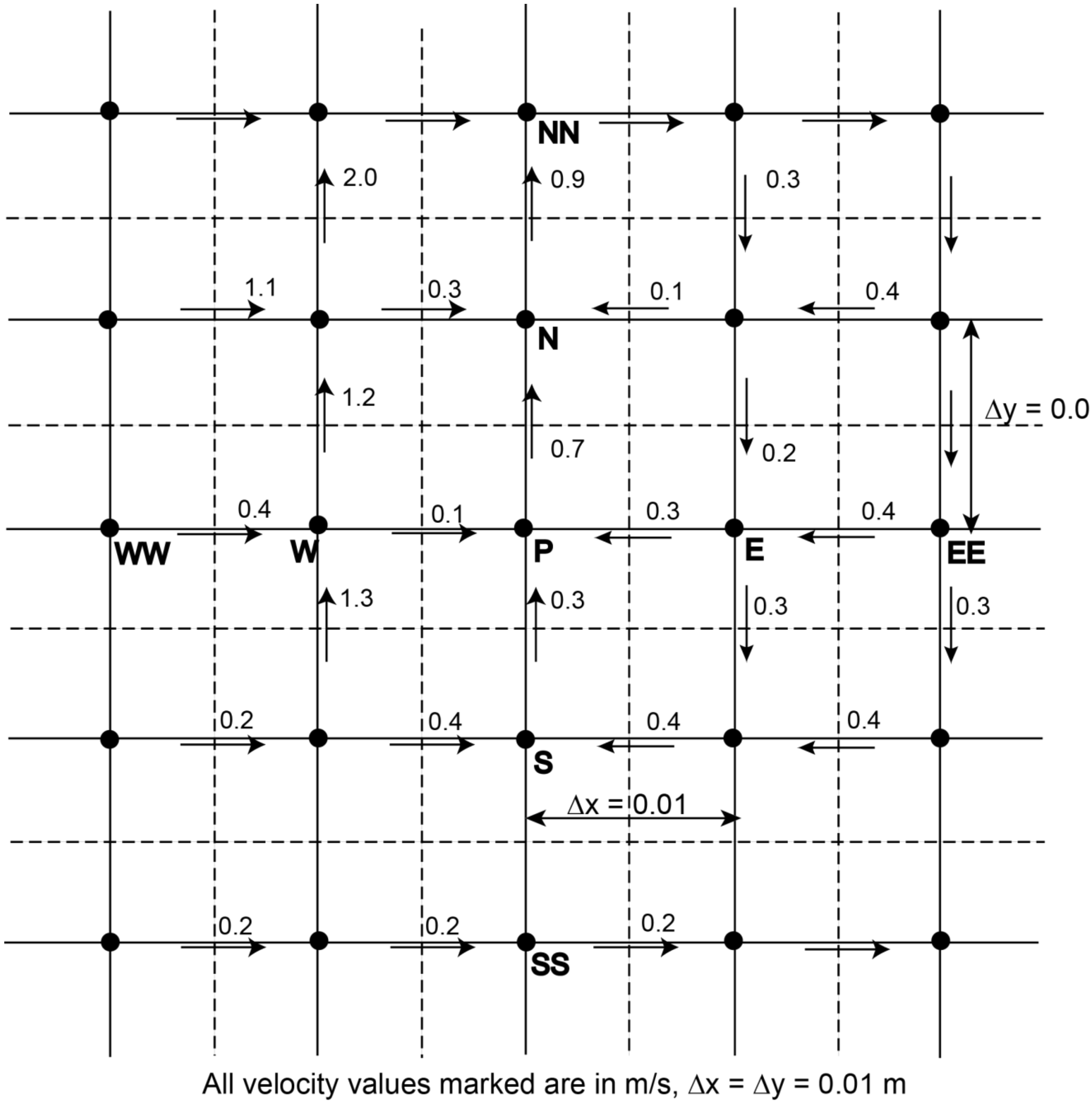

Figure Q.3

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USEFUL EQUATIONS

Note: All equations are in standard notations used during the lecture course.

Steady State One-dimensional Diffusion Equation is:

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0$$

where Γ is the diffusion coefficient, S is the source term.

Discretised form of the one-dimensional diffusion equation is:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u}$$

where

a_W	a_E	a_P
$\frac{\Gamma_w}{\delta x_{WP}}A_w$	$\frac{\Gamma_e}{\delta x_{PE}}A_e$	$a_W+a_E-S_P$

Boundary conditions can be introduced by cutting links with the appropriate face(s) and modifying the source terms. Equivalent expressions for 2-D and 3-D can be wrriten based on 1-D expressions. In conduction problems Γ is thermal conductivity k.

Steady State One-Dimensional Convection and Diffusion Equation

In the absence of sources, steady convection and diffusion of a property ϕ in a given one-dimensional flow field u is governed by:

$$\frac{d}{dx}(\rho u\phi)=\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)$$

The flow must also satisfy continuity $\frac{d(\rho u)}{dx}=0$

Discretised form of one-dimensional steady state convection diffusion equation is:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E}$$

with $\boxed{a_P = a_W + a_E + (F_e - F_w)}$

$$F_w=(\rho uA)_w,\qquad F_e=(\rho uA)_e$$

$$D_w=\frac{\Gamma_w}{\delta x_{WP}}A_w,\qquad D_e=\frac{\Gamma_e}{\delta x_{PE}}A_e$$

Coefficients depends on the discretisation scheme used.

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If source terms are present in the governing equation they could be accommodated using the standard practice.

The neighbour coefficients of the discretised equation for some common schemes are:

Scheme	$\boldsymbol{a_W}$	$\boldsymbol{a_E}$
Central differencing	$D_w + F_w/2$	$D_e - F_e/2$
Upwind differencing	$D_w + \max(F_w, 0)$	$D_e + \max(0, -F_e)$
Hybrid differencing	$\max[F_w, (D_w + F_w/2), 0]$	$\max[-F_e, (D_e - F_e/2), 0]$
Power law $Pe = \frac{F}{D} = \frac{\rho u}{\Gamma/\delta x}$	$D_w \max[0, (1 - 0.1\lvert Pe_w \rvert)^5] + \max(F_w, 0)$	$D_e \max[0, (1 - 0.1\lvert Pe_e \rvert)^5] + \max(-F_e, 0)$
Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.		

The discretised equation of the one-dimensional steady state convection diffusion equation using the standard QUICK scheme at a general internal node point is:

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + a_{WW} \phi_{WW} + a_{EE} \phi_{EE}$$

where $a_P = a_W + a_E + a_{WW} + a_{EE} + (F_e - F_w)$

The neighbour coefficients of the standard QUICK scheme in 1-D are:

	Standard QUICK
a_W	$D_w + \frac{6}{8}\alpha_w F_w + \frac{1}{8}\alpha_e F_e + \frac{3}{8}(1-\alpha_w)F_w$
a_{WW}	$-\frac{1}{8}\alpha_w F_w$
a_E	$D_e - \frac{3}{8}\alpha_e F_e - \frac{6}{8}(1-\alpha_e)F_e - \frac{1}{8}(1-\alpha_w)F_w$
a_{EE}	$\frac{1}{8}(1-\alpha_e)F_e$

with $\alpha_w = 1$ for $F_w > 0$ and $\alpha_e = 1$ for $F_e > 0$

$\alpha_w = 0$ for $F_w < 0$ and $\alpha_e = 0$ for $F_e < 0$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

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The SIMPLE algorithm

Pressure correction p' is the difference between correct pressure field p and the guessed pressure field p^*, so that

$$p = p^* + p'$$

Velocity corrections in two-dimensions u' and v' relate to the correct velocities u and v when the guessed velocities u^* and v^* as

$$u = u^* + u'$$

$$v = v^* + v'$$

Velocity corrections are obtained from pressure corrections field using

$$u_{i,J} = u^*_{i,J} + d_{i,J}(p'_{I-1.J} - p'_{I,J})$$

$$v_{I,j} = v^*_{I,j} + d_{I,j}(p'_{I,J-1} - p'_{I,J})$$

In a two-dimensional flow the continuity equation is:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

In a two-dimensional grid using west (W), east (E), south (S) and north (N) notations, the pressure correction equation derived from the continuity equation takes the form:

$$a_P p'_P = a_W p'_W + a_E p'_E + a_S p'_S + a_N p'_N + b'$$

where $a_{W`} = (\rho dA)_w$ ; $a_{E`} = (\rho dA)_e$; $a_{S`} = (\rho dA)_s$ ; $a_{N`} = (\rho dA)_n$

$$a_P = a_W + a_E + a_S + a_N$$

$$b' = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n$$

One-Dimensional Unsteady Heat Conduction is governed by:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + S$$

Discretised equation using the explicit scheme for one-dimensional unsteady heat conduction is

$$a_P T_P = a_W T_W^o + a_E T_E^o + \left[a_P^o - (a_W + a_E - S_p)\right] T_P^o + S_u \qquad (8.1)$$

where $a_P = a_P^o$

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and $$a_P^o = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

Discretised equation using the fully implicit scheme for one-dimensional unsteady heat conduction is

$$a_P T_P = a_W T_W + a_E T_E + a_P^o T_P^o + S_u$$

where $$a_P = a_P^o + a_W + a_E - S_p$$

and $$a_P^o = \rho c \frac{\Delta x}{\Delta t}$$

with

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

Equivalent expressions for 2-D and 3-D can be wrriten based on 1-D expressions.

Tri-Diagonal Matrix Algorithms (TDMA) for the Solution of Linear Equations

For a system of equations that has a tri-diagonal form any single equation may be written in the form:

$$-\beta_j \phi_{j-1} + D_j \phi_j - \alpha_j \phi_{j+1} = C_j$$

The solution can be obtained from the recurrence relationships:

$$\phi_j = A_j \phi_{j+1} + C'_j$$

where $$A_j = \frac{\alpha_j}{D_j - \beta_j A_{j-1}}$$

$$C'_j = \frac{\beta_j C'_{j-1} + C_j}{D_j - \beta_j A_{j-1}}$$

At the boundary points j = 1 and $j = n+1$ the values for A and C' are:

$$A_1 = 0 \text{ and } C'_1 = \phi_1 \quad \text{and} \quad A_{n+1} = 0 \text{ and } C'_{n+1} = \phi_{n+1}$$