Loughborough University

PROBABILITY THEORY
(24MAB170)

Semester 1 2024/2025 In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are **not** allowed).

Answer **THREE** questions.

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1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

   (a) State what it means for $\mathbb{P}$ to be a probability measure. [4]

   (b) Let $A, B \in \mathcal{F}$. Use the axioms of a probability space to prove the following identities:

   (i) $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B)$ [4]

   (ii) $\mathbb{P}((A \setminus B) \cup (B \setminus A)) = \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$ [4]

   (c) In a standard pack of 52 cards there are 26 red cards, 4 Aces, and exactly two of the Aces are red. A card is selected uniformly at random (each card being equally-likely to be chosen). Use your answer to part (b)(ii) to find the probability that the card is a red card or an Ace, *but not a red Ace*. [4]

   (d) A second card is selected uniformly at random from a complete pack. Find the conditional probability that the card is an Ace, given that you know the chosen card was red. [4]

2. (a) Suppose X and Y are jointly discrete random variables taking the values 0, -1 or 1 only, and that the joint probability mass function $p_{X,Y}(n, m)$ takes the values indicated in the table below:

		n		
	$p_{X,Y}(n,m)$	-1	0	1
	-1	0.1	0.1	0.1
m	0	0.1	0.2	0.1
	1	0.1	0.1	0.1

   (i) Verify that the values in the table define a proper joint probability mass function. [2]

   (ii) Find $\mathbb{P}(X > Y)$. [3]

   (iii) Find the marginal probability mass function for X. [3]

   (iv) Show that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$. [3]

   (v) Are X and Y independent? Justify your answer. [2]

   (b) Let X be a random variable with density function

$$f_X(x) = \begin{cases} x\mathrm{e}^{-x}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

   and let Y be a random variable independent of X, uniformly distributed on $[0, 1]$.

   (i) Find a joint density function for the random variables U, V, where

$$U = XY, \qquad V = X(1 - Y).$$ [5]

   (ii) Hence show that $\mathbb{P}(XY > c) = \mathrm{e}^{-c}$ for any constant $c > 0$. [2]

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3. (a) A random variable X has an exponential distribution with parameter $\lambda > 0$ if it has probability density function

$$f_X(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the moment-generating function of X is

$$\frac{\lambda}{\lambda - t}, \qquad t < \lambda. \tag{5}$$

(b) A random variable Y has the $\mathrm{Laplace}(\mu, b)$ distribution if its moment-generating function is

$$M_Y(t) = \frac{\mathrm{e}^{\mu t}}{1 - b^2 t^2}, \qquad |t| < \frac{1}{b}.$$

(i) Find the mean and variance of a $\mathrm{Laplace}(\mu, b)$ random variable. [6]

(ii) Show that if $\mu = 0$ all the odd moments of Y are zero. [3]

(c) If U and V are independent exponential random variables with parameters $\alpha, \beta > 0$ respectively then use moment-generating functions to show that $Z = \alpha U - \beta V$ is a Laplace random variable, and state the values of its parameters. [6]

4. (a) A pair of jointly continuous random variables have joint density function

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{3}\left(x^2 + \frac{xy}{3}\right), & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find a conditional density function $f_{X|Y}(x|y)$. [4]

(ii) Using the result of part (i) find the conditional expectation $\mathbb{E}(X \mid Y)$. [6]

(b) Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on earlier weather conditions. Suppose also that if it rains today then it will rain tomorrow with probability 0.6, and if it does not rain today then it will rain tomorrow with probability 0.3.

This can be modelled by a two state Markov chain on states 0 and 1 with transition probability matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$$

(i) To what weather conditions do states 0 and 1 correspond? [2]

(ii) Find the probability that it will rain two days from today, given that it is not raining today. [4]

(iii) Find the long-time limiting probabilities of rain and no rain. [4]

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