Loughborough University

Computational Fluid Dynamics 1

24WSC802

Semester 2, 2024-25 | In-Person Exam paper

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

Answer **ALL THREE** questions.

All questions carry equal marks.

You may take **TWO A4 sides of your own notes** into the examination venue.

Use of a calculator is permitted - It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are not allowed).

A range of formulae likely to be of benefit in the solution of these questions is provided at the rear of the paper.

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1. Fully-developed, laminar flow in a circular pipe satisfies the following axisymmetric momentum equation in cylindrical coordinates:

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = \frac{dP}{dz}$$

where V = axial fluid velocity, μ = dynamic viscosity of fluid, dP/dz = axial pressure gradient, r = local radius, z = axial coordinate.

The radius of the pipe is R = 6 mm . The fluid is glycerol with μ = 1.5 kg.m^{-1}.s^{-1}. The pressure gradient dP/dz = -3×10^4 Pa.m^{-1}.

a) Use the finite volume method in conjunction with a uniform radial grid of three control volumes between r = 0 and r = R to estimate the fluid velocity at the nodes of the control volumes. (Hint: you may find it helpful to multiply both sides of the axisymmetric momentum equation by r/μ prior to carrying out a 1-D control volume integration in the radial direction).

[15 marks]

b) Estimate the wall shear stress. Calculate the shear forces on a section of pipe with length of 1 m and demonstrate global conservation of z - momentum. [5 marks]

2. Shown in **Figure Q.2** is a two-dimensional cross section of a long beam. Heat is generated in this beam at a rate of 10 kW/m^3. The bottom surface of this beam is kept at a temperature of 400 °C while the top surface is at a temperature of 200°C. All other surfaces are exposed to convective heat transfer with a convective heat transfer coefficient of $h = 200$ W/(m^2.°C), and the ambient temperature is $T_\infty = 10$°C. The thermal conductivity of the material is 200 W/(m.°C). In usual notations conductive heat transfer in this situation is governed by:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + g = 0$$

The grid shown in **Figure Q.2** may be used to formulate discretised equations to calculate temperature distribution in this beam cross-section. Two-dimensional grid dimensions are $\Delta x = \Delta y$ = 0.1 m as shown in the **Figure Q.2**. The length of the beam may be considered to be infinitely long.

a) Write down the discretised form of the above equation for a general node P and provide expressions for its coefficients. [2 marks]

b) By considering symmetry, show how the general discretised equation could be modified at required nodes to obtain a set of equations to solve for temperature distribution. [9 marks]

c) Show how the set of equations could be arranged to use the TDMA method to solve for the temperature distribution. [2 marks]

d) Using the set of equations obtained in (c), tabulate the coefficients required for the application of the TDMA method in the form of **Table Q.2**. [4 marks]

e) Using the TDMA method solve for the temperature distribution at given nodes. [3 marks]

Attach Table Q.2 to the answer book.

TDMA method

For a set of equations of the form

$$-\beta_j \phi_{j-1} + D_j \phi_j - \alpha_j \phi_{j+1} = C_j$$

where β_j , α_j and $D_j$are coefficients, $j = 2,3,4 \dots\dots (n-1)$ are points along a line.

ϕ_j can be obtained from the recurrence formulae:

$$\phi_j = A_j \phi_{j+1} + C_j^{'}$$

$$A_j = \frac{\alpha_j}{(D_j - \beta_j A_{j-1})}$$

$$C_j^{'} = \frac{(\beta_j C_{j-1}^{'} + C_j)}{(D_j - \beta_j A_{j-1})}; \qquad A_1 = 0 \quad \text{and} \quad C_1' = \phi_1$$

3. In the absence of sources, steady one-dimensional convection and diffusion of a property ϕ in a given flow field u is governed by the transport equation:

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)$$

A one-dimensional uniform grid with nodal spacing Δx is used to discretise the above equation. A part of the grid is shown in **Figure Q.3**. The notation W and E are used to identify the west and east nodes of a general node P. The west and east cell faces are denoted by w and e, respectively. WW and EE may be used to identify the nodes two steps to the west and east of P, respectively. In the discretisation scheme known as the Linear Upwind Differencing (LUD), the cell face values are obtained by linear extrapolation from the two closest upstream neighboring nodes, i.e.,

$u_w > 0$ $\qquad\qquad \phi_w = \phi_W + \frac{\phi_W - \phi_{WW}}{2}$

$u_e > 0$ $\qquad\qquad \phi_e = \phi_P + \frac{\phi_P - \phi_W}{2}$

a) Use the above scheme for convective terms and central differencing for diffusion terms to derive the discretised form of the one-dimensional convection-diffusion equation given above for the case $u > 0$ everywhere. Using standard notation, provide expressions for its coefficients.You may use $F = \rho u$ and $D = \Gamma/\Delta x$ to simplify your expressions. [10 marks]

b) Rearrange the equation obtained in part (a) in the form

$a_P = \sum a_{nb}\phi_{nb} + S_u$ where *nb* stands for neighbour values and verify that $a_P = \sum a_{nb} + \alpha$ where α is the continuity imbalance. [2 marks]

If the flow field is given as $u = 2$ m/s and properties are given as $\rho = 1.0$ kg/m^3, $\Gamma = 0.1$ kg/(m.s), and grid spacing of the grid is $\Delta x = 0.1$ m.

c) Calculate the coefficients of the discretisation equation at a general node using (1) the scheme derived in part (a) and (2) the standard upwind scheme. [4 marks]

d) Comment on the properties of the two schemes, Linear Upwind Differencing and standard Upwind, making references to the values obtained in (c). [4 marks]

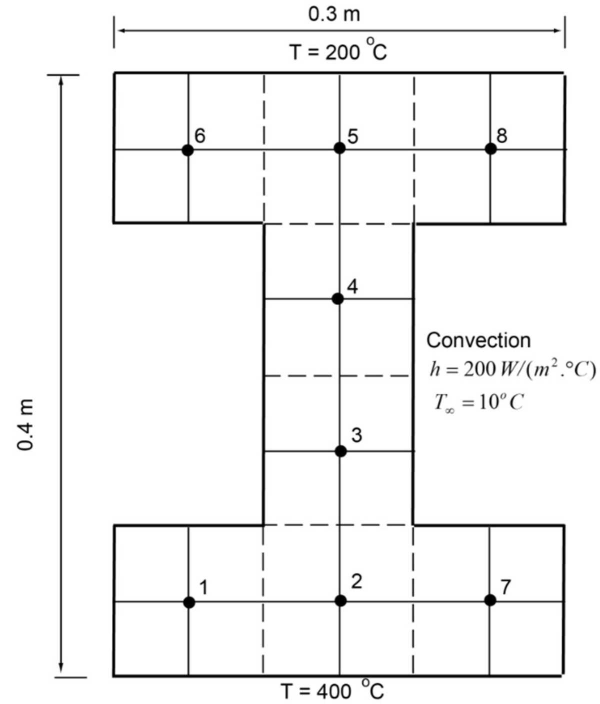

Figure Q.2

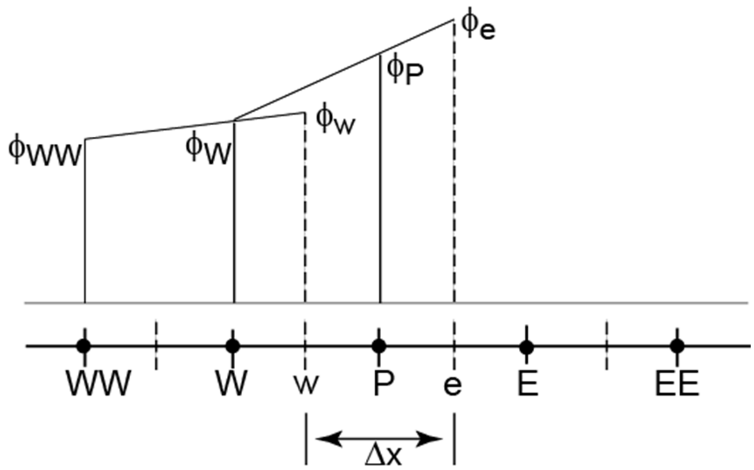

Figure Q.3

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Attach This to the Answer Book.

ID Number of the candidate: ___________________________

TABLE Q.2

Point	β_j	D_j	α_j	C_j	A_j	C_j'

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USEFUL EQUATIONS

Note: All equations are in standard notations used during the lecture course.

Steady State One-dimensional Diffusion Equation is:

$$\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)+S=0$$

where Γ is the diffusion coefficient, S is the source term.

Discretised form of the one-dimensional diffusion equation is:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u}$$

where

a_W	a_E	a_P
$\frac{\Gamma_w}{\delta x_{WP}}A_w$	$\frac{\Gamma_e}{\delta x_{PE}}A_e$	$a_W+a_E-S_P$

Boundary conditions can be introduced by cutting links with the appropriate face(s) and modifying the source terms. Equivalent expressions for 2-D and 3-D can be wrriten based on 1-D expressions. In conduction problems Γ is thermal conductivity k.

Steady State One-Dimensional Convection and Diffusion Equation

In the absence of sources, steady convection and diffusion of a property ϕ in a given one-dimensional flow field u is governed by:

$$\frac{d}{dx}(\rho u\phi)=\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)$$

The flow must also satisfy continuity $\frac{d(\rho u)}{dx}=0$

Discretised form of one-dimensional steady state convection diffusion equation is:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E}$$

with $\boxed{a_P=a_W+a_E+(F_e-F_w)}$

$$F_w=(\rho uA)_w, \qquad F_e=(\rho uA)_e$$

$$D_w=\frac{\Gamma_w}{\delta x_{WP}}A_w, \qquad D_e=\frac{\Gamma_e}{\delta x_{PE}}A_e$$

Coefficients depends on the discretisation scheme used.

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If source terms are present in the governing equation they could be accommodated using the standard practice.

The neighbour coefficients of the discretised equation for some common schemes are:

Scheme	a_W	a_E
Central differencing	$D_w + F_w/2$	$D_e - F_e/2$
Upwind differencing	$D_w + \max(F_w, 0)$	$D_e + \max(0, -F_e)$
Hybrid differencing	$\max[F_w, (D_w + F_w/2), 0]$	$\max[-F_e, (D_e - F_e/2), 0]$
Power law $Pe = \dfrac{F}{D} = \dfrac{\rho u}{\Gamma/\delta x}$	$D_w \max\left[0, (1 - 0.1\lvert Pe_w\rvert)^5\right] + \max(F_w, 0)$	$D_e \max\left[0, (1 - 0.1\lvert Pe_e\rvert)^5\right] + \max(-F_e, 0)$
Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.		

The discretised equation of the one-dimensional steady state convection diffusion equation using the standard QUICK scheme at a general internal node point is:

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + a_{WW} \phi_{WW} + a_{EE} \phi_{EE}$$

where $a_P = a_W + a_E + a_{WW} + a_{EE} + (F_e - F_w)$

The neighbour coefficients of the standard QUICK scheme in 1-D are:

	Standard QUICK
a_W	$D_w + \frac{6}{8}\alpha_w F_w + \frac{1}{8}\alpha_e F_e + \frac{3}{8}(1-\alpha_w)F_w$
a_{WW}	$-\frac{1}{8}\alpha_w F_w$
a_E	$D_e - \frac{3}{8}\alpha_e F_e - \frac{6}{8}(1-\alpha_e)F_e - \frac{1}{8}(1-\alpha_w)F_w$
a_{EE}	$\frac{1}{8}(1-\alpha_e)F_e$

with $\alpha_w = 1$ for $F_w > 0$ and $\alpha_e = 1$ for $F_e > 0$

$\alpha_w = 0$ for $F_w < 0$ and $\alpha_e = 0$ for $F_e < 0$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

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The SIMPLE algorithm

Pressure correction p' is the difference between correct pressure field p and the guessed pressure field $p*$, so that

$$p = p* + p'$$

Velocity corrections in two-dimensions u' and v' relate to the correct velocities u and v when the guessed velocities $u*$ and $v*$ as

$$u = u* + u'$$

$$v = v* + v'$$

Velocity corrections are obtained from pressure corrections field using

$$u_{i,J} = u^*_{i,J} + d_{i,J}(p'_{I-1,J} - p'_{I,J})$$

$$v_{I,j} = v^*_{I,j} + d_{I,j}(p'_{I,J-1} - p'_{I,J})$$

In a two-dimensional flow the continuity equation is:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

In a two-dimensional grid using west (W), east (E), south (S) and north (N) notations, the pressure correction equation derived from the continuity equation takes the form:

$$a_P p'_P = a_W p'_W + a_E p'_E + a_S p'_S + a_N p'_N + b'$$

where $a_{W`} = (\rho dA)_w$ ; $a_{E`} = (\rho dA)_e$; $a_{S`} = (\rho dA)_s$ ; $a_{N`} = (\rho dA)_n$

$$a_P = a_W + a_E + a_S + a_N$$

$$b' = (\rho u^* A)_w - (\rho u^* A)_e + (\rho v^* A)_s - (\rho v^* A)_n$$

One-Dimensional Unsteady Heat Conduction is governed by:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + S$$

Discretised equation using the explicit scheme for one-dimensional unsteady heat conduction is

$$a_P T_P = a_W T_W^o + a_E T_E^o + \left[a_P^o - (a_W + a_E - S_p)\right] T_P^o + S_u \qquad (8.1)$$

where $a_P = a_P^o$

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and $$a_P^o = \rho c \frac{\Delta x}{\Delta t}$$

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

Equivalent expressions for 2-D and 3-D can be written based on 1-D expressions.

Discretised equation using the fully implicit scheme for one-dimensional unsteady heat conduction is

$$a_P T_P = a_W T_W + a_E T_E + a_P^o T_P^o + S_u$$

where $$a_P = a_P^o + a_W + a_E - S_p$$

and $$a_P^o = \rho c \frac{\Delta x}{\Delta t}$$

with

a_W	a_E
$\frac{k_w}{\delta x_{WP}}$	$\frac{k_e}{\delta x_{PE}}$

Equivalent expressions for 2-D and 3-D can be wrriten based on 1-D expressions.

Tri-Diagonal Matrix Algorithms (TDMA) for the Solution of Linear Equations

For a system of equations that has a tri-diagonal form any single single equation may be written in the form:

$$-\beta_j \phi_{j-1} + D_j \phi_j - \alpha_j \phi_{j+1} = C_j$$

The solution can be obtained from the recurrence relationships:

$$\phi_j = A_j \phi_{j+1} + C'_j$$

where $$A_j = \frac{\alpha_j}{D_j - \beta_j A_{j-1}}$$

$$C'_j = \frac{\beta_j C'_{j-1} + C_j}{D_j - \beta_j A_{j-1}}$$

At the boundary points $j = 1$ and $j = n+1$ the values for A and C' are:

$$A_1 = 0 \text{ and } C'_1 = \phi_1 \text{ and } A_{n+1} = 0 \text{ and } C'_{n+1} = \phi_{n+1}$$