Loughborough University

PROBABILITY THEORY
(25MAB170)

Semester 1 25/26 — 2 hours

This examination is to take place in-person at a central University venue under exam conditions. The standard length of time for this paper is **2 hours**.

You will not be able to leave the exam hall for the first 30 or final 15 minutes of your exam. Your invigilator will collect your exam paper when you have finished.

Help during the exam

Invigilators are not able to answer queries about the content of your exam paper. Instead, please make a note of your query in your answer script to be considered during the marking process.

If you feel unwell, please raise your hand so that an invigilator can assist you.

You may use a calculator for this exam. It must comply with the University's Calculator Policy for In-Person exams, in particular that it must not be able to transmit or receive information (e.g. mobile devices and smart watches are **not** allowed).

Answer **THREE** questions.

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1. (a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

   (i) Define what it means for $\mathcal{F}$ to be a σ-algebra. [3]

   (ii) Suppose $\Omega = \{A, B, C\}$. Does the collection

   $$\mathcal{G} = \{\emptyset, \Omega, \{A\}, \{B, C\}\}$$

   form a σ-algebra on Ω? Justify your answer. [4]

   (b) In a certain species of animal, it is suggested that the probability mass function for the number X of offspring is of the form

   $$p_X(0) = \lambda,$$
   $$p_X(n) = \alpha p^{n-1}, \qquad n = 1, 2, 3, \ldots$$

   for some constants $0 < \lambda < 1$, $0 < p < 1$ and α a real constant.

   (i) For this to be a proper probability mass function show that

   $$\alpha = (1 - \lambda)(1 - p). \quad [4]$$

   (ii) Find the probability that such an animal has fewer than 5 offspring. [5]

   (iii) Find the expected number $\mathbb{E}(X)$ of offspring. [Hint: You may make use of the summation formula

   $$\sum_{k=1}^{\infty} k a^k = \frac{a}{(1-a)^2}, \qquad |a| < 1.] \quad [4]$$

2. (a) A pair of jointly continuous random variables X and Y have joint probability density function

   $$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}\sin(x + y), & 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

   (i) Show that a marginal probability density function for X is

   $$f_X(x) = \begin{cases} \frac{1}{2}(\sin x + \cos x), & 0 \le x \le \frac{\pi}{2}, \\ 0, & \text{otherwise.} \end{cases} \quad [5]$$

   (ii) Write down a marginal probability density function for Y. [2]

   (iii) Are X and Y independent? Justify your answer. [3]

   (iv) Find the probability $\mathbb{P}(X > Y)$. [4]

   (b) The radius R of a (spherical) ball-bearing made by a machine is a continuous random variable with probability density function

   $$f_R(r) = 140 r^3 (1 - r)^3, \qquad 0 < r < 1.$$

   Find a probability density for the volume of the ball-bearings $V = 4\pi R^3/3$. [6]

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3. (a) Let X and Y be random variables with moment-generating functions $M_X(t)$ and $M_Y(t)$ respectively.

(i) Show that the moment-generating function for aX, with a constant is $M_X(at)$. [2]

(ii) If X and Y are independent show that the moment-generating function for $X+Y$ is $M_X(t)M_Y(t)$. [2]

(b) A sequence of independent, identically distributed random variables X_i, $i = 1, 2, \ldots$ have probability density function

$$f(x) = \begin{cases} \frac{1}{2c}, & |x| \leq c, \\ 0, & \text{otherwise,} \end{cases}$$

where $c > 0$ is a parameter.

(i) Find the moment-generating function of the random variables. [4]

(ii) Use your answer (i) to show that the mean is 0, and find a choice of the parameter c so that the variance is equal to 1. [5]

(iii) With the choice of parameter c from part (ii), consider the random variable

$$Z = \frac{X_1 + \cdots + X_n}{\sqrt{n}}.$$

Using standard properties of moment-generating functions, show that the moment-generating function of Z can be written as

$$M_Z(t) = \left(\frac{\mathrm{e}^{\sqrt{3/nt}} - \mathrm{e}^{-\sqrt{3/nt}}}{2\sqrt{3/nt}} \right)^n. \quad [3]$$

(iv) Calculate $\lim_{n\to\infty} M_Z(t)$ and interpret your answer. (Hint: Use logarithms.) [4]

4. (a) A pair of jointly continuous random variables have joint density function

$$f_{X,Y}(x,y) = \begin{cases} 60x^2y(x-y)^2, & 0 \leq x \leq 1,\ 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find a conditional density function $f_{X|Y}(x|y)$. [4]

(ii) Using the result of part (i) find the conditional expectation $\mathbb{E}(X \mid Y)$. [6]

(b) A Markov chain on four states $\{1, 2, 3, 4\}$ has matrix of transition probabilities

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(i) Calculate $\mathbb{P}(X_2 = 4 \mid X_0 = 2)$ and $\mathbb{P}(X_2 = 4 \mid X_0 = 1)$. [5]

(ii) Given that you are told the Markov chain is ergodic, find the long-time limiting probabilities to be in each of the four states. [5]

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