1. Taylor Series

If a function $f(x)$ is differentiable up to *nth* order in some neighbourhood of a point $x = a$, then in the neighbourhood, the function can be represented by a power series in $(x - a)$.

$$f(x) = f(a) + (x - a)\, f^{(1)}(a) \ldots + \frac{(x - a)^r}{r!} f^{(r)}(a) + \ldots$$

$$+ \frac{(x - a)^{n-1}}{(n - 1)!} f^{(n-1)}(a) + R_n$$

For many functions the remainder term $R_n \to 0$ as $n \to \infty$, and so we can write

$$f(x) = \sum_{r=0}^{\infty} \frac{(x - a)^r}{r!} f^{(r)}(a) \qquad \textbf{(1.1)}$$

This series expansion is known as the Taylor series. For the case of $a = 0$ it is known as the Maclaurin's series.

Some important series easily derived are for the exponential, cosine and sine functions.

$$exp\,(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{r!} \qquad \textbf{(1.2)}$$

$$cos\,(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad \textbf{(1.3)}$$

$$sin\,(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \textbf{(1.4)}$$

Series expansions are widely used in engineering to provide approximations to functions. Sufficient accuracy is often achieved by taking just the first two terms of the series. If $(x - a) << 1$ then higher powers will contribute little.

Say we wish to express the value of a function for small displacements h from an operating point a. Then $h = x - a$ and equation (1.1) becomes

$$f(a + h) = \sum_{r=0}^{\infty} \frac{h^r}{r!} f^{(r)}(a)$$

Taking just the first two terms (sometimes referred to as linearisation) gives

$$f(a+h) \approx f(a) + hf'(a)$$

If we take as an example $f(x) = x^n$, then

$$(a+h)^n \approx a^n + na^{n-1}h$$

and it is clear that all powers of h greater than 1 have been discarded. Provided $h << 1$ such approximations are reasonably accurate.