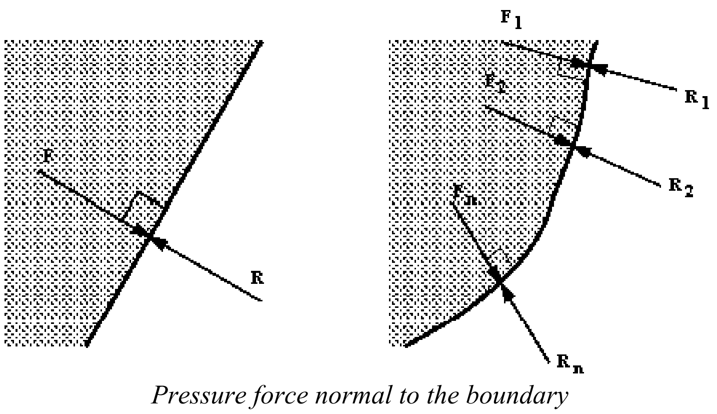
2. Forces in Static Fluids

[This material relates predominantly to modules ELP034, ELP035]

2.1 Fluids Statics

The general rules of statics (as applied in solid mechanics) apply to fluids at rest. From earlier we know that:

- a static fluid can have **no shearing force** acting on it, and that
- any force between the fluid and the boundary must be acting at right angles to the boundary.



Pressure force normal to the boundary

Note that this statement is also true for curved surfaces, in this case the force acting at any point is normal to the surface at that point. The statement is also true for any imaginary plane in a static fluid. We use this fact in our analysis by considering elements of fluid bounded by imaginary planes.
We also know that:

- For an element of fluid at rest, the element will be in equilibrium - the sum of the components of forces in any direction will be zero.

- The sum of the moments of forces on the element about any point must also be zero.

It is common to test equilibrium by resolving forces along three mutually perpendicular axes and also by taking moments in three mutually perpendicular planes an to equate these to zero.

2.2 Pressure

As mentioned above a fluid will exert a normal force on any boundary it is in contact with. Since these boundaries may be large and the force may differ from place to place it is convenient to work in terms of pressure, *p*, which is the force per unit area.

If the force exerted on each unit area of a boundary is the same, the pressure is said to be *uniform*.

$$\text{pressure} = \frac{\text{Force}}{\text{Area over which the force is applied}}$$

$$p = \frac{F}{A}$$

Units: Newton's per square metre, Nm^{-2}, $kgm^{-1}s^{-2}$.

(The same unit is also known as a Pascal, *Pa*, i.e. $1Pa = 1\ Nm^{-2}$)

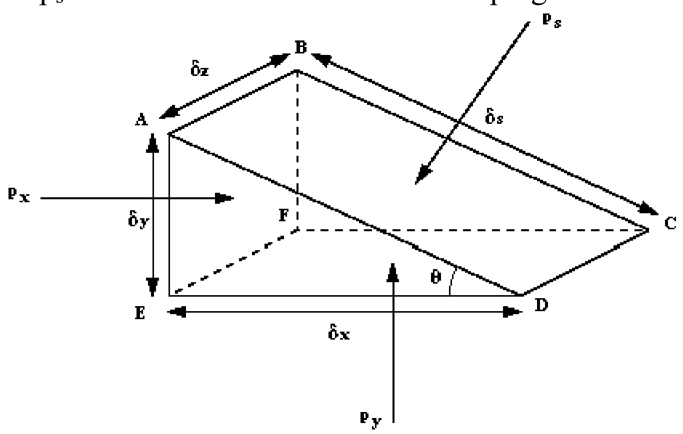
(Also frequently used is the alternative SI unit the *bar,* where $1bar = 10^5\ Nm^{-2}$)

Dimensions: $ML^{-1}T^{-2}$

2.3 Pascal's Law for pressure at a point

(Proof that pressure acts equally in all directions. Only need to study if you have to be convinced or are interested. Otherwise, jump to ✸)

By considering a small element of fluid in the form of a triangular prism which contains a point P, we can establish a relationship between the three pressures p_x in the x direction, p_y in the y direction and p_s in the direction normal to the sloping face.



Triangular prismatic element of fluid

The fluid is a rest, so we know there are no shearing forces, and we know that all force are acting at right angles to the surfaces .i.e.

p_s acts perpendicular to surface ABCD,

p_x acts perpendicular to surface ABFE and

p_y acts perpendicular to surface FECD.

And, as the fluid is at rest, in equilibrium, the sum of the forces in any direction is zero.

Summing forces in the x-direction:

Force due to p_x,

$$F_{x_x} = p_x \times Area_{ABFE} = p_x \delta x \, \delta y$$

Component of force in the x-direction due to p_s,

$$\begin{aligned} F_{x_s} &= -p_s \times Area_{ABCD} \times \sin\theta \\ &= -p_s \delta s \, \delta z \frac{\delta y}{\delta s} \\ &= -p_s \delta y \, \delta z \end{aligned}$$

($\sin\theta = \delta y / \delta s$)

Component of force in x-direction due to p_y,

$$F_{x_y} = 0$$

To be at rest (in equilibrium)

$$\begin{aligned} F_{x_x} + F_{x_s} + F_{x_y} &= 0 \\ p_x \delta x \delta y + (-p_s \delta y \delta z) &= 0 \\ p_x &= p_s \end{aligned}$$

Similarly, summing forces in the y-direction. Force due to p_y,

$$F_{y_y} = p_y \times Area_{EFCD} = p_y \delta x \delta z$$

Component of force due to p_s,

$$\begin{aligned} F_{y_s} &= -p_s \times Area_{ABCD} \times \cos\theta \\ &= -p_s \delta s \delta z \frac{\delta x}{\delta s} \\ &= -p_s \delta x \delta z \end{aligned}$$

($\cos\theta = {}^{\delta x}\!/_{\delta s}$)

Component of force due to p_x,

$$F_{y_x} = 0$$

Force due to gravity,

$$\begin{aligned}\text{weight} &= -\text{specific weight} \times \text{volume of element}\\ &= -\rho g \times \frac{1}{2}\delta x\,\delta y\,\delta z\end{aligned}$$

To be at rest (in equilibrium)

$$F_{y_y} + F_{y_s} + F_{y_x} + \text{weight} = 0$$

$$p_y \delta x \delta y + \left(-p_s \delta x \delta z\right) + \left(-\rho g \frac{1}{2}\delta x \delta y \delta z\right) = 0$$

The element is small i.e. δx, δy and δz are small, and so $\delta x \delta y \delta z$ is <u>very</u> small and considered negligible, hence

$$p_y = p_s$$

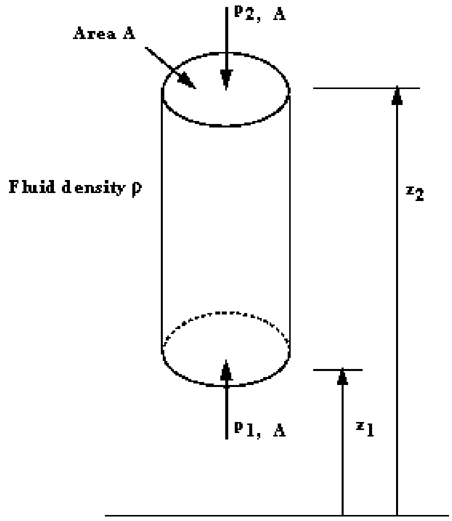
thus

$$p_x = p_y = p_s$$

Considering the prismatic element again, p_s is the pressure on a plane at any angle θ, the x, y and z directions could be any orientation. The element is so small that it can be considered a point so the derived expression $p_x = p_y = p_s$. indicates that pressure at any point is the same in all directions.
(The proof may be extended to include the z axis).

Pressure at any point is the same in all directions.
This is known as Pascal's Law and applies to fluids at rest.

2.4 Variation of pressure vertically in a fluid under gravity



Vertical elemental cylinder of fluid

In the above figure we can see an element of fluid which is a vertical column of constant cross sectional area, A, surrounded by the same fluid of mass density ρ. The pressure at the bottom of the cylinder is p_1 at level z_1, and at the top is p_2 at level z_2. The fluid is at rest and in equilibrium so all the forces in the vertical direction sum to zero. i.e. we have

$$\begin{aligned}
\text{Force due to } p_1 \text{ on A (upward)} &= p_1 A \\
\text{Force due to } p_2 \text{ on A (downward)} &= p_2 A \\
\text{Force due to weight of element (downward)} &= mg \\
&= \text{mass density} \times \text{volume} = \rho g A (z_2 - z_1)
\end{aligned}$$

Taking upward as positive, in equilibrium we have

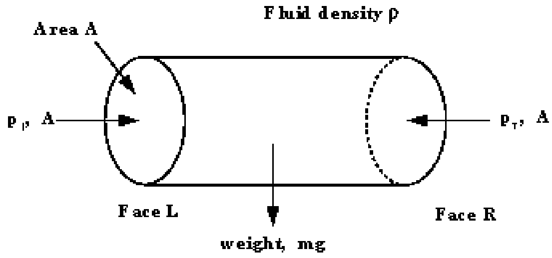
$$p_1 A - p_2 A - \rho g A (z_2 - z_1) = 0$$

$$p_2 - p_1 = -\rho g (z_2 - z_1)$$

Thus in a fluid under gravity, pressure decreases with increase in height $z = (z_2 - z_1)$.

2.5 Equality of pressure at the same level in a static fluid

Consider the horizontal cylindrical element of fluid in the figure below, with cross-sectional area A, in a fluid of density ρ, pressure p_1 at the left hand end and pressure p_2 at the right hand end.



Horizontal elemental cylinder of fluid

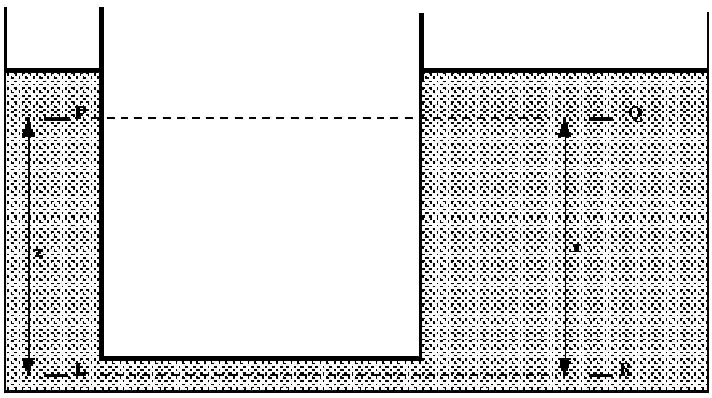
The fluid is at equilibrium so the sum of the forces acting in the x direction is zero.

$$p_l A = p_r A$$

$$p_l = p_r$$

Pressure in the horizontal direction is constant.

This result is the same for any *continuous* fluid. It is still true for two connected tanks which appear not to have any direct connection, for example consider the tank in the figure below.



Two tanks of different cross-section connected by a pipe

We have shown above that $p_l = p_r$ and from the equation for a vertical pressure change we have

$$p_l = p_p + \rho g z$$

and

$$p_r = p_q + \rho g z$$

so

$$p_p + \rho g z = p_q + \rho g z$$
$$p_p = p_q$$

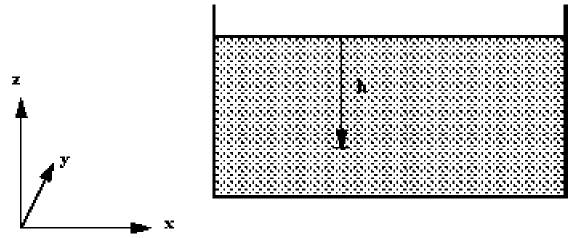
This shows that the pressures at the two equal levels, P and Q are the same.

2.6 Pressure and Head

In a static fluid of constant density we have the relationship $\frac{dp}{dz} = -\rho g$, as shown above. This can be integrated to give

$$p = -\rho g z + \text{constant}$$

In a liquid with a free surface the pressure at any depth *z* measured from the free surface so that *z* = -*h* (see the figure below)



Fluid head measurement in a tank.

This gives the pressure

$$p = \rho g h + \text{constant}$$

At the surface of fluids we are normally concerned with, the pressure is the atmospheric pressure, $p_{atmospheric}$. So

$$p = \rho g h + p_{atmospheric}$$

As we live constantly under the pressure of the atmosphere, and everything else exists under this pressure, it is convenient (and often done) to take atmospheric pressure as the datum. So we quote pressure as above or below atmospheric.
Pressure quoted in this way is known as gauge pressure i.e.

Gauge pressure is

$$p_{gauge} = \rho g h$$

The lower limit of any pressure is zero - that is the pressure in a perfect vacuum. Pressure measured above this datum is known as absolute pressure i.e.

Absolute pressure is

$$p_{absolute} = \rho g h + p_{atmospheric}$$
$$\text{Absolute pressure} = \text{Gauge prssure} + \text{Atmospheric pressure}$$

As *g* is (approximately) constant, the gauge pressure can be given by stating the vertical height of any fluid of density ρ which is equal to this pressure.

$$p = \rho g h$$

This vertical height is known as **head** of fluid.

Note: If pressure is quoted in *head*, the density of the fluid *must* also be given.

2.7 Fluid Pressure on a surface

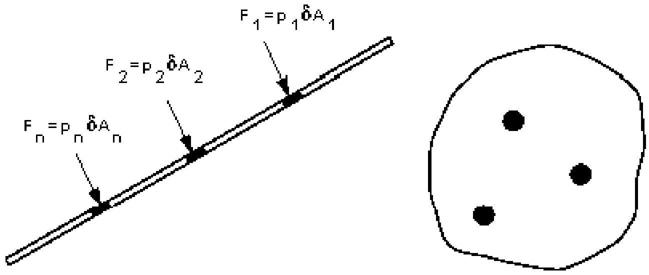
Pressure is defined as force per unit area. If a pressure p acts on a small area δA then the force exerted on that area will be

$$F = p\,\delta A$$

Since the fluid is at rest the force will act at right-angles to the surface.

General submerged plane

Consider the plane surface shown in the figure below. The total area is made up of many elemental areas. The force on each elemental area is always normal to the surface but, in general, each force is of different magnitude as the pressure usually varies.



We can find the total or ***resultant*** force, R, on the plane by summing up all of the forces on the small elements i.e.

$$R = p_1\delta A_1 + p_2\delta A_2 + \ldots\ldots p_n\delta A_n = \sum p\,\delta A$$

This resultant force will act through the centre of pressure, hence we can say

If the surface is a plane the force can be represented by one single resultant force, acting at right-angles to the plane through the centre of pressure.

Horizontal submerged plane

For a horizontal plane submerged in a liquid (or a plane experiencing uniform pressure over its surface), the pressure, p, will be equal at all points of the surface. Thus the resultant force will be given by

$$R = \text{pressure} \times \text{area of plane}$$
$$R = pA$$

Curved submerged surface

If the surface is curved, each elemental force will be a different magnitude and in different direction but still normal to the surface of that element. The resultant force can be found by resolving all forces into orthogonal co-ordinate directions to obtain its magnitude and direction. This will **always** be less than the sum of the individual forces, $\sum p\,\delta A$.

2.8 Worked Example

<u>1. Example of the pressure and head relationship</u>:

What is a pressure of 500 *kNm*$^{-2}$

A) In head of water of density, ρ = 1000 kgm^{-3}
***Hint*: Use $p = \rho gh$,**

$$h = \frac{p}{\rho g} = \frac{500 \times 10^3}{1000 \times 9.81} = 50.95m \text{ of water}$$

B) In head of Mercury density ρ = 13.6 x 10^3 kgm^{-3}.

$$h = \frac{500 \times 10^3}{13.6 \times 10^3 \times 9.81} = 3.75m \text{ of Mercury}$$

C) In head of a fluid with relative density γ = 8.7.
(remember $\rho = \gamma \times \rho_{water}$)

$$h = \frac{500 \times 10^3}{(8.7 \times 1000) \times 9.81} = 5.86m \text{ of fluid } \gamma = 8.7$$