6. The Momentum Equation

[This material relates predominantly to modules ELP034, ELP035]

6.1 Definition of the momentum equation

Applications of the momentum equation:

6.2 The force due to the flow around a pipe bend.
6.3 Impact of a jet on a plane
6.4 Force on a curved Vane

6.1 Definition of The Momentum Equation

We have all seen moving fluids exerting forces. The lift force on an aircraft is exerted by the air moving over the wing. A jet of water from a hose exerts a force on whatever it hits. In fluid mechanics the analysis of motion is performed in the same way as in solid mechanics - by use of Newton's laws of motion. Account is also taken for the special properties of fluids when in motion.

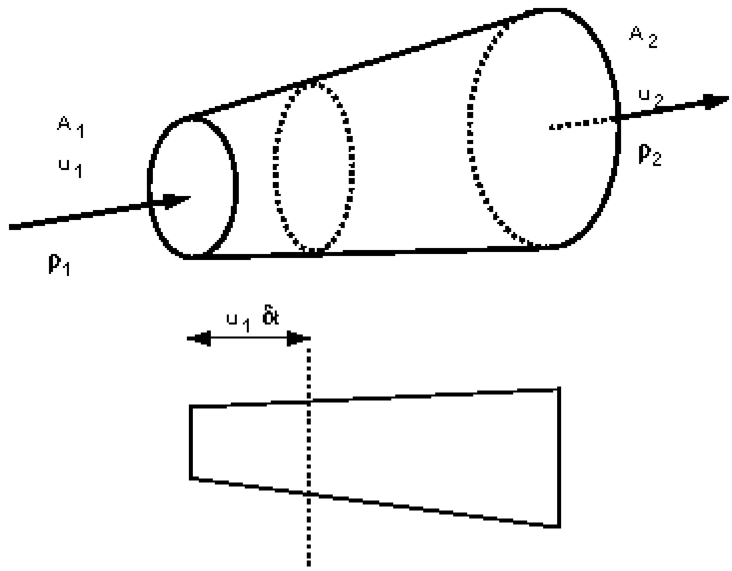
The momentum equation is a statement of Newton's Second Law and relates the sum of the forces acting on an element of fluid to its acceleration or rate of change of momentum. You will probably recognise the equation $F = ma$ which is used in the analysis of solid mechanics to relate applied force to acceleration. In fluid mechanics it is not clear what mass of moving fluid we should use so we use a different form of the equation.

Newton's 2nd Law can be written:

> *The Rate of change of momentum of a body is equal to the resultant force acting on the body, and takes place in the direction of the force.*

To determine the rate of change of momentum for a fluid we will consider a streamtube as we did for the Bernoulli equation,

We start by assuming that we have *steady* flow which is *non-uniform* flowing in a stream tube.



A streamtube in three and two-dimensions

In time δt a volume of the fluid moves from the inlet a distance $u\delta t$, so the volume entering the streamtube in the time δt is

$$\text{volume entering the stream tube} = \text{area} \times \text{distance} = A_1 u_1 \delta t$$

this has mass,

$$\text{mass entering stream tube} = \text{volume} \times \text{density} = \rho_1 A_1 u_1 \delta t$$

and momentum

$$\text{momentum of fluid entering stream tube} = \text{mass} \times \text{velocity} = \rho_1 A_1 u_1 \delta t\, u_1$$

Similarly, at the exit, we can obtain an expression for the momentum leaving the steamtube:

$$\text{momentum of fluid leaving stream tube} = \rho_2 A_2 u_2 \delta t\, u_2$$

We can now calculate the force exerted by the fluid using Newton's 2nd Law. The force is equal to the rate of change of momentum. So

$$\text{Force} = \text{rate of change of momentum}$$

$$F = \frac{(\rho_2 A_2 u_2 \delta t\, u_2 - \rho_1 A_1 u_1 \delta t\, u_1)}{\delta t}$$

We know from continuity that $Q = A_1 u_1 = A_2 u_2$, and if we have a fluid of constant density, i.e. $\rho_1 = \rho_2 = \rho$, then we can write

$$F = Q\rho(u_2 - u_1)$$

For an alternative derivation of the same expression, as we know from conservation of mass in a stream tube that

$$\text{mass into face 1} = \text{mass out of face 2}$$

we can write

$$\text{rate of change of mass} = \dot{m} = \frac{dm}{dt} = \rho_1 A_1 u_1 = \rho_2 A_2 u_2$$

The rate at which momentum leaves face 1 is

$$\rho_2 A_2 u_2 u_2 = \dot{m} u_2$$

The rate at which momentum enters face 2 is

$$\rho_1 A_1 u_1 u_1 = \dot{m} u_1$$

Thus the rate at which momentum changes across the stream tube is

$$\rho_2 A_2 u_2 u_2 - \rho_1 A_1 u_1 u_1 = \dot{m} u_2 - \dot{m} u_1$$

i.e.

$$\text{Force} = \text{rate of change of momentum}$$

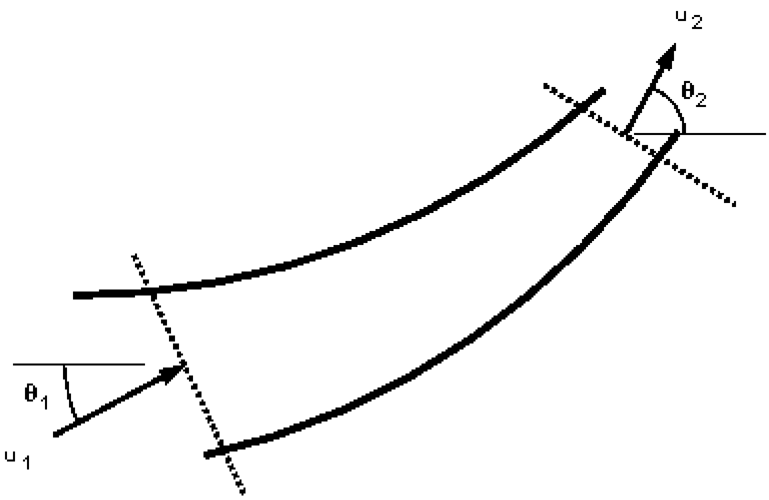
$$F = \dot{m}(u_2 - u_1)$$

$$F = Q\rho(u_2 - u_1)$$

This force is acting in the direction of the flow of the fluid.

This analysis assumed that the inlet and outlet velocities were in the same direction - i.e. a one dimensional system. What happens when this is not the case?

Consider the two dimensional system in the figure below:



Two dimensional flow in a streamtube

At the inlet the velocity vector, u_1, makes an angle, θ_1, with the x-axis, while at the outlet u_2 make an angle θ_2. In this case we consider the forces by resolving in the directions of the co-ordinate axes.

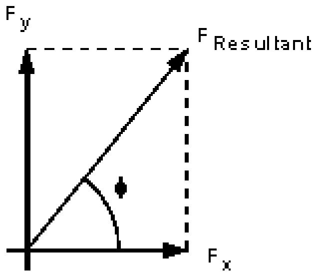
The force in the x-direction

$$\begin{aligned} F_x &= \text{Rate of change of momentum in x-direction} \\ &= \text{Rate of change of mass} \times \text{change in velocity in x-direction} \\ &= \dot{m}\left(u_2 \cos\theta_2 - u_1 \cos\theta_1\right) \\ &= \dot{m}\left(u_{2_x} - u_{1_x}\right) \\ &= \rho Q\left(u_2 \cos\theta_2 - u_1 \cos\theta_1\right) \\ &= \rho Q\left(u_{2_x} - u_{1_x}\right) \end{aligned}$$

And the force in the y-direction

$$\begin{aligned} F_y &= \dot{m}\left(u_2 \sin\theta_2 - u_1 \sin\theta_1\right) \\ &= \dot{m}\left(u_{2_y} - u_{1_y}\right) \\ &= \rho Q\left(u_2 \sin\theta_2 - u_1 \sin\theta_1\right) \\ &= \rho Q\left(u_{2_y} - u_{1_y}\right) \end{aligned}$$

We then find the **resultant force** by combining these vectorially:



$$F_{\text{resultant}} = \sqrt{F_x^2 + F_y^2}$$

And the angle which this force acts at is given by

$$\phi = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

For a three-dimensional (x, y, z) system we then have an extra force to calculate and resolve in the z-direction. This is considered in exactly the same way.

In summary we can say:

The total force **exerted on** the fluid = rate of change of momentum through the control volume

$$F = \dot{m}(u_{\text{out}} - u_{\text{in}})$$
$$= \rho Q(u_{\text{out}} - u_{\text{in}})$$

Remember that we are working with vectors so F is in the direction of the velocity. This force is made up of three components:

F_R = Force exerted on the fluid by any solid body touching the control volume

F_B = Force exerted on the fluid body (e.g. gravity)

F_P = Force exerted on the fluid by fluid pressure outside the control volume

So we say that the total force, F_T, is given by the sum of these forces:

$$F_T = F_R + F_B + F_P$$

The force exerted **by** the fluid **on** the solid body touching the control volume is opposite to F_R . So the reaction force, R, is given by

$$R = -F_R$$

Applications of the Momentum Equation

We will consider the following examples:

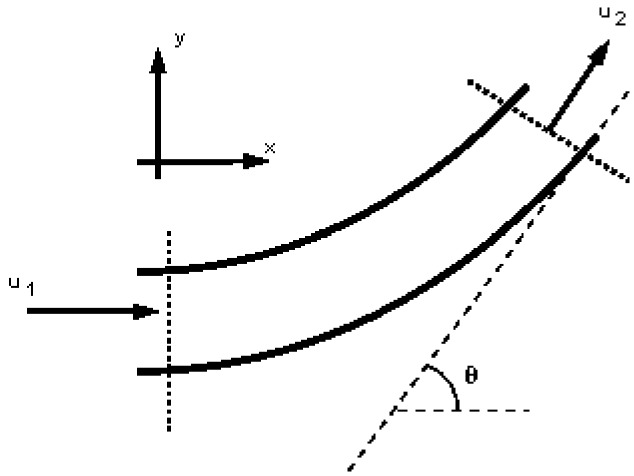
a) Force due to the flow of fluid round a pipe bend.

b) Impact of a jet on a plane surface.

c) Force due to flow round a curved vane.

6.2 The force due the flow around a pipe bend

Consider a pipe bend with a constant cross section lying in the horizontal plane and turning through an angle of θ° .



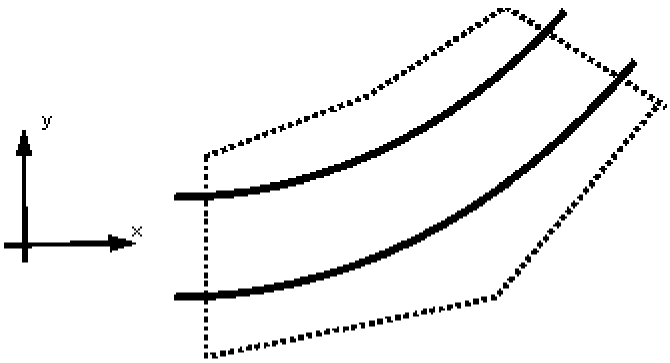
Flow round a pipe bend of constant cross-section

Why do we want to know the forces here? Because the fluid changes direction, a force (very large in the case of water supply pipes,) will act in the bend. If the bend is not fixed it will move and eventually break at the joints. We need to know how much force a support (thrust block) must withstand.

Step in Analysis:

1. Draw a control volume
2. Decide on co-ordinate axis system
3. Calculate the **total** force
4. Calculate the **pressure** force
5. Calculate the **body** force
6. Calculate the **resultant** force

1. Control Volume



The control volume is draw in the above figure, with faces at the inlet and outlet of the bend and encompassing the pipe walls.

2 Co-ordinate axis system

It is convenient to choose the co-ordinate axis so that one is pointing in the direction of the inlet velocity. In the above figure the x-axis points in the direction of the inlet velocity.

3 Calculate the **total** force

In the x-direction:

$$F_{T_x} = \rho Q\left(u_{2_x} - u_{1_x}\right)$$
$$u_{1_x} = u_1$$
$$u_{2_x} = u_2 \cos\theta$$
$$F_{T_x} = \rho Q\left(u_2 \cos\theta - u_1\right)$$

In the y-direction:

$$F_{T_y} = \rho Q\left(u_{2_y} - u_{1_y}\right)$$
$$u_{1_y} = u_1 \sin 0 = 0$$
$$u_{2_y} = u_2 \sin\theta$$
$$F_{T_y} = \rho Q u_2 \sin\theta$$

4 Calculate the **pressure** force

$$F_P = \text{pressure force at 1 - pressure force at 2}$$
$$F_{P_x} = p_1 A_1 \cos 0 - p_2 A_2 \cos\theta = p_1 A_1 - p_2 A_2 \cos\theta$$
$$F_{P_y} = p_1 A_1 \sin 0 - p_2 A_2 \sin\theta = -p_2 A_2 \sin\theta$$

5 Calculate the **body** force

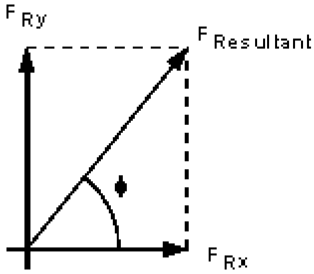
There are no body forces in the x or y directions. The only body force is that exerted by gravity (which acts into the paper in this example - a direction we do not need to consider).

6 Calculate the **resultant** force

$$F_{T_x} = F_{R_x} + F_{P_x} + F_{B_x}$$
$$F_{T_y} = F_{R_y} + F_{P_y} + F_{B_y}$$

$$F_{R_x} = F_{T_x} - F_{P_x} - 0 = \rho Q\left(u_2 \cos\theta - u_1\right) - p_1 A_1 + p_2 A_2 \cos\theta$$
$$F_{R_y} = F_{T_y} - F_{P_y} - 0 = \rho Q u_2 \sin\theta + p_2 A_2 \sin\theta$$

And the resultant force **on the fluid** is given by



$$F_R = \sqrt{F_{R\,x}^2 - F_{R\,y}^2}$$

And the direction of application is

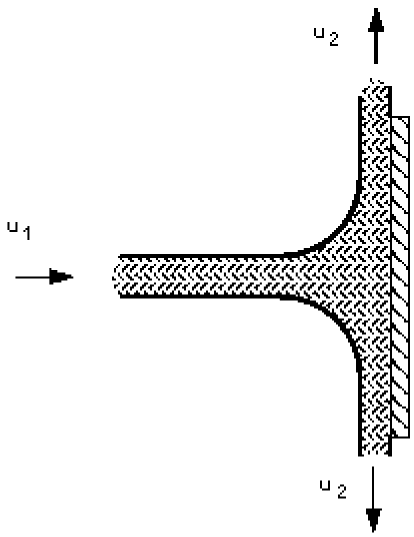
$$\phi = \tan^{-1}\left(\frac{F_{R_y}}{F_{R_x}}\right)$$

the force **on the bend** is the same magnitude but in the opposite direction

$$R = -F_R$$

6.3 Impact of a Jet on a Plane

We will first consider a jet hitting a flat plate (a plane) at an angle of 90°, as shown in the figure below.
We want to find the reaction force of the plate i.e. the force the plate will have to apply to stay in the same position.

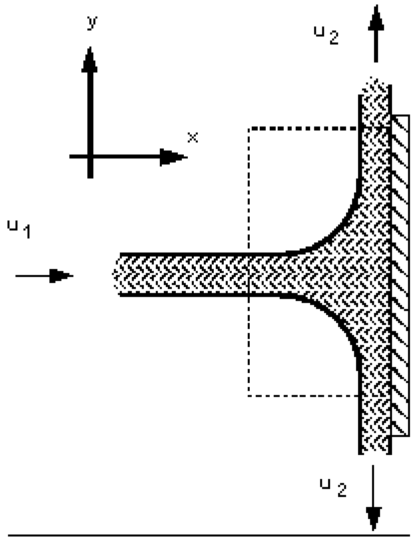


A perpendicular jet hitting a plane.

The analysis take the same procedure as above:

1. Draw a control volume
2. Decide on co-ordinate axis system
3. Calculate the **total** force
4. Calculate the **pressure** force
5. Calculate the **body** force
6. Calculate the **resultant** force

1 & 2 Control volume and Co-ordinate axis are shown in the figure below.



3 Calculate the **total** force

$$F_{T_x} = \rho Q(u_{2_x} - u_{1_x})$$
$$= -\rho Q u_{1_x}$$

As the system is symmetrical the forces in the y-direction cancel i.e.

$$F_{T_y} = 0$$

4 Calculate the **pressure force.**

The pressure force is zero as the pressure at both the inlet and the outlets to the control volume are atmospheric.

5 Calculate the **body** force

As the control volume is small we can ignore the body force due to the weight of gravity.

6 Calculate the **resultant** force

$$F_{T_x} = F_{R_x} + F_{P_x} + F_{B_x}$$
$$F_{R_x} = F_{T_x} - 0 - 0$$
$$= -\rho Q u_{1_x}$$

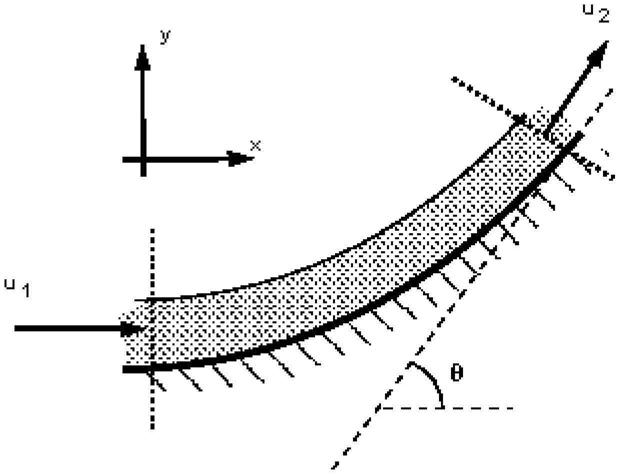
Exerted **on the fluid.**

The force **on the plane** is the same magnitude but in the opposite direction

$$R = -F_{R_x}$$

6.4 Force on a curved vane

This case is similar to that of a pipe, but the analysis is simpler because the pressures are equal - atmospheric , and both the cross-section and velocities (in the direction of flow) remain constant. The jet, vane and co-ordinate direction are arranged as in the figure below.



Jet deflected by a curved vane.

1 & 2 Control volume and Co-ordinate axis are shown in the figure above.

3 Calculate the **total** force in the x direction

$$F_{T_x} = \rho Q(u_2 - u_1 \cos\theta)$$

but $u_1 = u_2 = \dfrac{Q}{A}$, so

$$F_{T_x} = -\rho \frac{Q^2}{A}(1 - \cos\theta)$$

and in the y-direction

$$F_{T_y} = \rho Q(u_2 \sin\theta - 0)$$
$$= \rho \frac{Q^2}{A}$$

4 Calculate the **pressure force.**

Again, the pressure force is zero as the pressure at both the inlet and the outlets to the control volume are atmospheric.

5 Calculate the **body** force

No body forces in the x-direction, $F_{B_x} = 0$.

In the y-direction the body force acting is the weight of the fluid. If V is the volume of the fluid on he vane then,

$$F_{B_x} = \rho g V$$

(This is often small is the jet volume is small and sometimes ignored in analysis.)

6 Calculate the **resultant** force

$$F_{T_x} = F_{R_x} + F_{P_x} + F_{B_x}$$
$$F_{R_x} = F_{T_x}$$

$$F_{T_y} = F_{R_y} + F_{P_y} + F_{B_y}$$
$$F_{R_y} = F_{T_y}$$

And the resultant force **on the fluid** is given by

$$F_R = \sqrt{F_{R\,x}^2 - F_{R\,y}^2}$$

And the direction of application is

$$\phi = \tan^{-1}\left(\frac{F_{R\,y}}{F_{R\,x}}\right)$$

exerted on the fluid.

The force **on the vane** is the same magnitude but in the opposite direction