2. Complex Numbers

The complex number $z = x + iy$ can also be written in modulus argument form

$$z = r\cos\theta + i\,r\sin\theta \qquad \textbf{(2.1)}$$

The real part, $Re\ z = x = r\cos\theta$ and the imaginary part, $1mz = y = r\sin\theta$. Complex numbers can be shown as points on an Argand diagram.

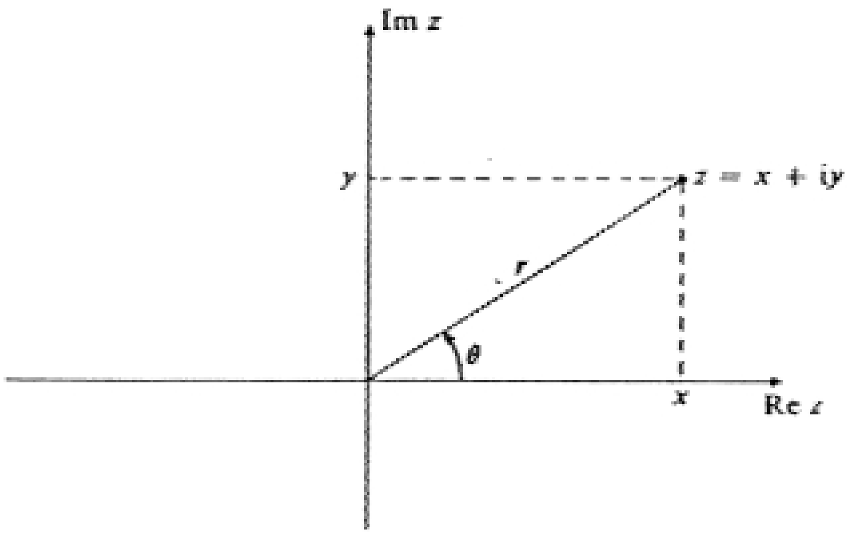

Fig 1.1 *An Argand Diagram*

Standard rules exist defining, addition, subtraction, multiplication and divisions of complex numbers.

We can show that there exists the following convenient alternative form of equation (2.1):

$$z = r\exp(i\theta) \qquad \textbf{(2.2)}$$

Just as a series expansion for *exp (x)* was calculated, (equation (1.2)), so we can write down the series for *exp (iθ)* as

Taking the real and imaginary parts we get:

$$exp(i\theta) = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \ldots$$

$$Re\,[\,exp\,(i\theta\,)] = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$

and

$$Im\,[\,exp\,(i\theta\,)] = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

Comparison with equations (1.3) and (1.4) gives us that

$$Re\,[\,exp\,(i\theta\,)] = cos\,\theta$$

and

$$Im\,[\,exp\,(i\theta\,)] = sin\,\theta$$

so that

$$exp\,(i\theta\,) = cos\,\theta + i\,sin\,\theta \qquad \textbf{(2.3)}$$

and finally

$$r\,exp\,(i\theta\,) = r\,cos\,\theta + ir\,sin\,\theta$$

as required.

Changing θ to $-\theta$ in (2.3) gives

$$exp\,(-i\theta\,) = cos\,\theta - i\,sin\,\theta$$

From (2.3) and (2.4) we can write

$$cos\,\theta = \frac{1}{2}\,[\,exp\,(i\theta\,) + exp\,(-i\theta\,)]$$

$$sin\,\theta = \frac{1}{2i}\,[\,exp\,(i\theta\,) - exp\,(-i\theta\,)]$$

From (2.2) complex numbers are straightforward to multiply:

$$r_1 \exp(i\theta_1)\, r_2 \exp(i\theta_2) = r_1 r_2 \exp[i(\theta_1+\theta_2)]$$

By extension, for z with unit modulus,

$$z^n = [\exp(i\theta)]^n = \exp(in\theta) \qquad \textbf{(2.5)}$$

This is de Moivre's theorem. It is often written in the alternative form:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Equation 2.2 also provides a very useful framework for the analysis of periodic systems. For example, if $\theta = \omega t$, where ω is an angular velocity, then the function

$$f(t) = R\exp(i\omega t) = R\cos\omega t + iR\sin\omega t$$

has real and imaginary parts which undergo sinusoidal variations with amplitude R and period $2\pi/\omega$ (although out of phase by $\pi/2$).

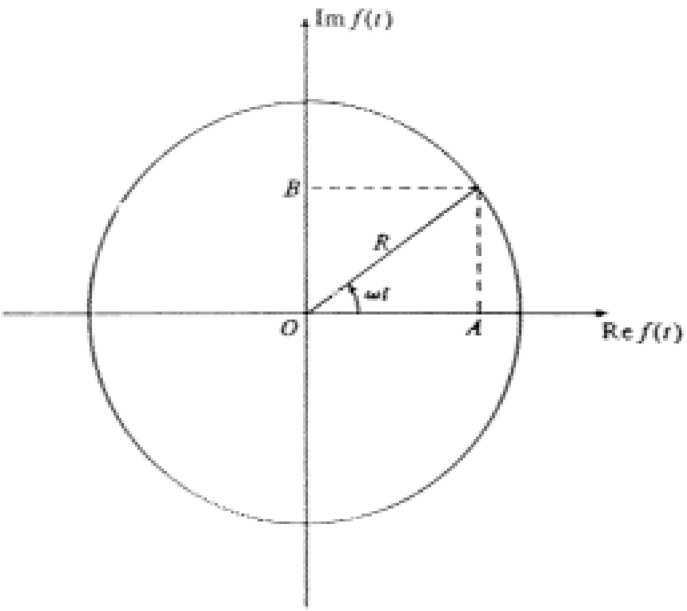

Fig 1.2 *The representation of exp(iωt) in an Argand Diagram*