3. Differential Equations and the Laplace Transform

Some definitions:

Ordinary differential equations	-	have only one independent variable.
Order	-	order of highest derivative which appears.
Degree	-	exponent of the highest derivative when all exponents are integers.
Linear	-	all derivatives plus the dependent variable *y* are linear and multiplied at most by a function of *x*.
Boundary (or initial) conditions	-	additional relationships which enable the arbitrary constants of the solution to be fixed. NB. There will be *n* such constants for an *nth*- order equation.

3.1 First-order equations

The general first-order equation can be written as

$$Q(x, y)\frac{dy}{dx} + P(x, y) = 0 \qquad \textbf{(3.1)}$$

If the equation can be rewritten in the form

$$p(x)\, dx + q(y)\, dy = 0$$

the variables are said to be separated and the solution is given trivially as

$$\int^{x} p(x_1)\, dx_1 + \int^{y} q(y_1)\, dy_1 = c$$

Generally this will not be the case. The solution sought can be written as

$$c = f(x, y)$$

Differentiating this with respect to *x* gives

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad \textbf{(3.2)}$$

By comparison with the original equation (3.1) we get

$$\frac{\partial f}{\partial x} / \frac{\partial f}{\partial y} = P/Q$$

If the original equation (3.1) pre-multiplied by a function *g(x, y)* is to result in the form given by (3.2) then

$$\frac{\partial f}{\partial y} = gQ \quad and \quad \frac{\partial f}{\partial x} = gP$$

since

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

we get

$$\frac{\partial}{\partial y}(gP) = \frac{\partial}{\partial x}(gQ) \qquad \textbf{(3.3)}$$

For the particular case of *g (x, y)* constant the original differential equation is known as exact and the solution can be written down as

$$c = f(x, y) = \int^{x} P(x_1, y)\, dx_1 + h(y) \qquad \textbf{(3.4)}$$

where the function $h(y)$ is chosen so that

$$\frac{\partial}{\partial y}\left[\int^{x} P(x_1, y)\, dx_1\right] + \frac{dh}{dy} = Q \qquad \textbf{(3.5)}$$

Alternatively the solution

$$c = \int^{y} Q(x, y_1)\, dy_1 + k(x)$$

where

$$\frac{\partial}{\partial x}\left[\int^{y} Q(x, y_1)\, dy_1\right] + \frac{dk}{dx} = P$$

may be easier to evaluate.

If the equation is not exact, an integrating factor $g(x, y)$ must be found. Expanding equation (3.3) gives

$$g\left(\frac{\partial p}{\partial y} - \frac{\partial Q}{dx}\right) = Q\frac{\partial g}{\partial x} - P\frac{\partial g}{\partial y}$$

If P is substituted from the original equation we get an equation for g:

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{1}{g}\frac{dg}{dx}$$

For common occurring cases where the LHS of (3.5) is a function of x only the solution can

$$g(x) = \exp\left[\int^{x} \frac{1}{Q(x_1, y)}\left(\frac{\partial P(x_1, y)}{\partial y} - \frac{\partial Q(x_1, y)}{\partial x_1}\right) dx_1\right]$$

be obtained directly by integration as

3.2 Second-order equations

Second-order equations occur frequently in the description of dynamical systems with springs, damping and inertial forces. Very often these will also be linear. The general case of importance is

$$f_2(x)\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_o\, y(x) = f(x) \qquad \textbf{(3.6)}$$

where, in dynamic analysis, $f(x)$ on the RHS is the forcing function. A special case has $f(x) = 0$. A solution of (3.6) is known as the particular function, whilst a solution of (3.6) with $f(x) = 0$ is called the complementary function. The general solution of (3.6) is the sum of these two.

Solutions of (3.6) are often of the form

$$y(x) = A\exp(\lambda_1 x) + B\exp(\lambda_2 x) + c$$

where λ_1 , λ_2 and c are fixed numbers and A and B the two arbitrary constants to be fixed through application of the boundary conditions.

For many dynamic systems (3.6) can be further simplified so that f_0, f_1 and f_2 become constants, ie

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0\, y = f(x) \qquad \textbf{(3.7)}$$

If $f(x)$ is zero or takes a simple form we can proceed as follows.

Assume the complementary function is of the form

$$y_2(x) = A \exp(\lambda x)$$

This results in

$$(a_2 \lambda^2 + a_1 \lambda + a_0) A \exp(\lambda x) = 0$$

which is clearly satisfied if

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \qquad \textbf{(3.8)}$$

This expression, known as the auxiliary equation, is solved to give λ (two solutions in general) λ_1 and λ_2

$$\lambda_{\frac{1}{2}} = \frac{-a_1 \pm (a_1^2 - 4 a_2 a_0)^{\frac{1}{2}}}{2 a_2}$$

Since the differential equation (3.7) with $f(x) = 0$ is linear we can write down the complementary function (in this case the general solution for $f(x) = 0$) as

$$y_2(x) = A \exp(\lambda_1 x) + B \exp(\lambda_2 x)$$

3.3 The Laplace transform

The Laplace transform is an integral transform given by

$$L(f) \equiv F(s) \equiv \int_0^\infty exp\,(-sx)\; f(x)\, dx$$

It is a linear transformation which takes x to a new, in general, complex variable s. It is used to convert differential equations into purely algebraic equations.

Some examples are given in the table below:

$f(x)$	$L(f)$
1	s^{-1}
x^n	$n!\, s^{-(n+1)}$
$sin\,(bx)$	$b\,(s^2 + b^2)^{-1}$
$cos\,(bx)$	$s\,(s^2 + b^2)^{-1}$
$exp\,(ax)$	$(s - a)^{-1}$
$x^n\, exp\,(ax)$	$n!\,(s - a)^{-(n+1)}$
$x^{1/2}$	$1/2\,(\pi\, s^{-3})^{1/2}$
$x^{-1/2}$	$(\pi/s)^{1/2}$
$U\,(x - x_0) = 1,\ x \geq x_0$ $= 0,\ x < x_0$	$s^{-1}\, exp\,(-sx_0)$

Deriving the inverse transform is problematic. It tends to be done through the use of tables of transforms such as the one above.

For application to differential equations we start by evaluating $L\left(\frac{df}{dx}\right)$ using integration by parts, the result is

$$L\left(\frac{df}{dx}\right) = sL(f) - f(0) \qquad \textbf{(3.11)}$$

This process can be repeated for $\frac{d^2 f}{dx^2}, \frac{d^3 f}{dx^3}$ etc, giving

$$L(\, f^{(n)}\,) = s^n\, L(f) - \sum_{r=0}^{n-1} s^{n-r-1}\, f^{(r)}\,(0) \qquad \textbf{(3.12)}$$

Hence the Laplace transform of any derivative can be expressed in terms of $L(f)$ plus derivatives evaluated at $x = 0$. It is thus possible to rewrite any differential equation in terms of an algebraic equation for $L(y)$.

Four useful properties of Laplace transforms can be established.

i) $L\,[\, f_1\,(x) \pm f_2\,(x)\,] = F_1\,(s) \pm F_2\,(s)$

ii) $L[\,k\,f(x)\,] = k\,F(s)$

iii) $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ (final value theorem)

iv) $L[\,f(x - X)\,] = e^{-sX} F(s)$ (shifting theorem)

3.3.1 Control theory applications

Laplace transforms are widely used in classical control theory. The independent variable is often taken as time t. In many applications the transformed differential equation will turn out to be the ratio of two polynomials in s, for example

$$F(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \dots a_0}{s^n + b_{n-1} s^{n-1} + \dots b_0}$$

If required this can be split by partial fractions to give

$$F(s) = F_1(s) + F_2(s) + \dots$$

where the inverse transform L^{-1} can be found separately for each element. Hence

$$f(t) = L^{-1}[\,F_1(s)] + L^{-1}[\,F_2(s)] + \dots$$

Often in control theory the inverse transform is not taken since much can be concluded (regarding stability etc) from the form of $F(s)$.

The transfer function of a linear system is defined as the ratio of the Laplace transform of the output of the system to the Laplace transform of the input to the system. It is denoted by $G(s)$ or $H(s)$. The linear assumption means that the properties of the system (eg $G(s)$) are not dependent on the state of the system (value of t or s).

For transfer functions, the initial conditions are assumed to be zero so that

$$L(\,f^{(n)}\,) = s^n\,L(f)$$

Differential equations are transformed into the Laplace domain simply by replacing $\frac{d}{dt}$ by s, $\frac{d^2}{dt^2}$ by s^2 etc. The resulting transfer function can then be written

$$G(s) = \frac{P(s)}{Q(s)}$$

where P and Q are polynomials in s. The properties of G are determined by the roots of $Q(s) = 0$ (which is known as the characteristic equation).

NB. These roots are referred to as poles of the system since $G(s)$ becomes infinite at these values of s. The order of the system is the order of $Q(s)$

Example 1 Spring

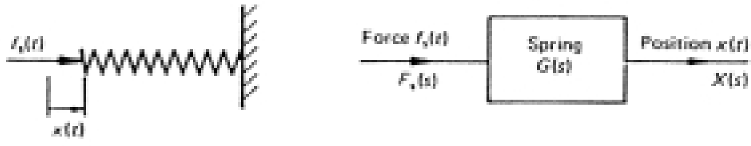

Fig. Simple spring

$f_s(t) = Kx$ so $F_s(s) = KX(s)$

The transfer function *G(s)* is given then by

$$\frac{X(s)}{F_s(s)} = \frac{1}{K}$$

Example 2 Damper

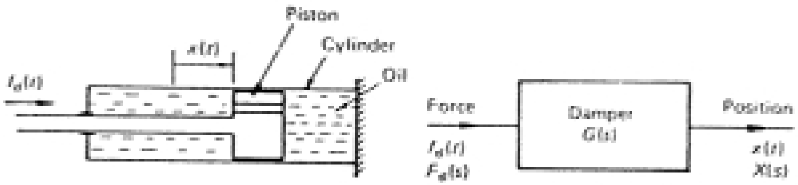

Fig. Hydraulic damper

$$f_d(t) = C\frac{dx(t)}{dt}$$

Assuming $x(0) = 0$, the Laplace transform gives

$$F_d(s) = Cs\,X(s)$$

Hence $$G(s) = \frac{X(s)}{F_d(s)} = \frac{1}{Cs}$$

Example 3 Mass-Spring-Damper

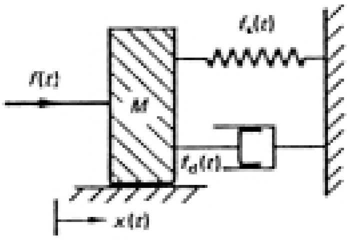

Fig. Mass-spring-damper

$$f(t) = M\ddot{x} + C\dot{x} + Kx$$

For zero initial conditions

$$F(s) = Ms^2\, X(s) + Cs\, X(s) + KX(s)$$

and so the transfer function is

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + K}$$

Often the differential equation is written as

$$\ddot{x} + 2\zeta\, \omega_n\, \dot{x} + \omega_n^2\, x = \omega_n^2\, u(t) \qquad \textbf{(3.13)}$$

where $u(t)$ is the forcing term, $\omega_n = \sqrt{\left(\frac{K}{M}\right)}$ is the system natural frequency and ζ is known as the damping factor. In these terms the transfer function is written as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\, \omega_n\, s + \omega_n^2}$$

3.3.2 Calculating transient response to a step input

The approach developed above can be used to calculate the transient response of linear systems. First order systems are straightforward to analyse but not particularly interesting. We will concentrate here on second order systems.

The output of a system is often denoted by *c(t)* and its Laplace transform by *C(s)* with the input being *u(t)* and *U(s)* respectively.

$$C(s) = U(s)\,G(s)$$

$$= \frac{U(s)\,\omega_n^2}{s^2 + 2\zeta\,\omega_{ns} + \omega_n^2}$$

The Laplace transform of a step input is $\frac{1}{s}$ and so

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\,\omega_n s + \omega_n^2)}$$

$$= \frac{1}{s} + \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2}$$

using partial fractions where A_1 and A_2 are constants and p_1 and p_2 are the roots of the characteristic equation.

Hence, taking inverse transforms,

$$c(t) = 1 + A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

where $A_1 = -\frac{1}{2} - \frac{\zeta}{2\sqrt{(\zeta^2 - 1)}}$ and $A_2 = -\frac{1}{2} + \frac{\zeta}{2\sqrt{(\zeta^2 - 1)}}$

and $p_1 = -\zeta\,\omega_n + \omega_n\sqrt{(\zeta^2 - 1)}$ and $p_2 = -\zeta\,\omega_r - \omega_n\sqrt{(\zeta^2 - 1)}$

The nature of the response depends on the nature of the roots, which is determined by the damping factor ζ. The figure below shows the form of step response for a second order system.