5. Fourier Transform and Spectral Density Functions

Harmonic functions (written as $e^{i\omega t}$ or $\cos(\omega t)$) are attractive from an analytical point of view for a number of reasons: they are straightforward to differentiate, integrate and multiply; moduli are easily evaluated and each contains only one frequency. The last property will be found to be useful in the context of analysing systems which are characterised by a fundamental (resonance) frequency. Complexity can be dealt with through superposition.

Technically, the harmonic functions can be shown to the mutually orthogonal and complete (any reasonable function can be expressed as a linear combination of them).

If the period of $f(t)$ is T, then mutual orthogonality means

$$\int^{T} f_n(t)\,.\,f_m(t)\,dt = 0 \quad n \neq m$$

If we take $f(t) = \cos(2\pi nt/T)$ with integer n, it can be shown to be orthogonal by integrating over the range $(-T/2, T/2)$.

Since $\cos(2\pi nt/T)$ is an even function, linear combinations will always be even.

A more general set of functions is then required to give completeness. As a function can be expressed as the sum of an even and an odd part, the suggested set of functions is:

$$f_{n,m}(t) = \cos(2\pi nt/T) + \sin(2\pi mt/T)$$

for integers n and m.

It is straightforward to show that the orthogonal property still applies.

A particular linear combination of the above set (which also has the desired properties) is

$$\cos(2\pi nt/T) + i\sin(2\pi nt/T)$$

This is attractive in that it can be written as simply $\exp(i\,2\pi nt/T)$.

A Fourier series is an expansion of a function in terms of its Fourier components as suggested by the above discussion.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n \exp(in\omega t) \qquad (5.1)$$

The constants C_n are known as the Fourier coefficients, $\omega = 2\pi/T$ is the fundamental frequency.

If a function $f(t)$ is known, the Fourier components can be projected out by pre-multiplying by $\exp(-im\,\omega t)$ and integrating from $-T/2$ to $T/2$.

Fourier series can be differentiated and integrated term by term.

5.1 Fourier transforms

The Fourier transform (or Fourier integral) is obtained formally be allowing the period T of the Fourier series to become infinite. Instead of only integer frequency contributions (as in the Fourier series) a continuous function of frequency results. From equation (5.1) we can write down

$$f(t) = K_1 \int_{-\infty}^{\infty} g(\omega) \exp(i\omega t)\, d\omega \qquad (5.2)$$

where K_1 is an arbitrary constant. By analogy, the Fourier coefficients C_n become a continuous function $g(\omega)$, ie

$$g(\omega) = K_2 \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt \qquad (5.3)$$

K_2 is also arbitrary, although the product $K_1 K_2$ is fixed. It is conventional to take K_1 and K_2 as $1/\sqrt{2\pi}$.

Equation (5.3) is known as the Fourier transform and (5.2) the inverse Fourier transform. The function $g(\omega)$ is called the spectrum of $f(t)$.

It can be noted that as an integral transform it is similar to the Laplace transform but that both limits of integration are infinite.

Example 1: Fourier transform of exponential decay function $f(t) = 0$ for $t < 0$ and $f(t) = \exp(-\lambda t)$ for $t \geq 0$ with $\lambda > 0$.

$$g(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-\lambda t) \exp(-i\omega t)\, dt$$

$$= (2\pi)^{-1/2} (\lambda + i\omega)^{-1}$$

NB: Complex contour integration is required to derive this, however it is easily checked using the inverse transform.

Example 2: Fourier transform of Gaussian or normal distribution function, (zero mean and standard deviation σ).

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{t^2}{2\sigma^2} \right) \qquad -\infty < t < \infty$$

$$thus \quad g(\omega) = (2\pi\sigma)^{-1} \int_{-\infty}^{\infty} exp(-t^2/2\sigma^2)\, exp(-i\omega t)\, dt$$

$$= (2\pi\sigma)^{-1} \int_{-\infty}^{\infty} exp\left[-\frac{1}{2\sigma^2}(t^2 + 2\sigma^2 i\omega t + (\sigma^2 i\omega)^2 - (\sigma^2 i\omega)^2) \right] dt$$

which can be rewritten as

$$g(\omega) = \frac{exp(-\sigma^2\omega^2/2)}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} exp\left[-\frac{(t + i\sigma^2\omega)^2}{2\sigma^2} \right] dt$$

Finally this can be shown to give

$$g(\omega) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{\sigma^2\omega^2}{2} \right)$$

which is also a Gaussian distribution with zero mean but with standard deviation equal to $1/\sigma$.

ie $\sigma_\omega \sigma_t = 1$

The narrower in time an impulse is, the greater the spread of frequency components.

5.2 Properties of the Fourier transform

These are listed without proof:

i) Differentiation $FT(f^1(t)) = i\,\omega\, g(\omega)$

ii) Integration $FT(\int^t f(s)\, ds) = -\, i\, \omega^{-1} g(\omega) + 2\pi\, C\, \delta(\omega)$

iii) Translation $FT\,(f\,(t + a)) = exp\,(i\,a\,\omega)\; g\,(\omega)$

iv) Exponential multiplication $FT\,(exp\,(\alpha\,t)\,f\,(t)) = g(\,\omega + i\,\alpha)$

where α may be real, imaginary or complex and $2\pi\,C\delta\,(\omega)$ is the Fourier transform of the constant of integration associated with the indefinite integral, and $\delta(\omega)$ is the delta-function given by

$$\delta(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} exp\,(i\omega t)\, d\omega$$

5.3 Properties of the Dirac delta-function

Combining the definitions of the Fourier transform and the inverse Fourier transform we can write (suitably arranged)

$$f(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \times (2\pi)^{-1/2} \int_{-\infty}^{\infty} dt' \exp(-i\omega t')\, f(t')$$

By swapping the integration order: $f(t) = \int_{-\infty}^{\infty} d't\; f(t') \times (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \exp\,[i\omega\,(t - t')]$

This is very instructive since the second term, which in the delta-function notation can be written simply as $\delta\,(t - t')$, has the property of selecting out just one point from the infinite integral and assigning a finite value to the result. This leads to an alternative definition of the delta-function, namely

$$\delta\,(t - t') = 0 \;\forall\; t, t'$$

except for $t = t'$ where the function is infinite or more rigorously

$$\int_{-\infty}^{\infty} \delta(t - t')\, d't = 1$$

5.4 Parseval's theorem

Parseval's theorem can be stated as:

$$\int_{-\infty}^{\infty} |\, f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\, g(\omega)|^2 \, d\omega$$

Its proof regimes use the properties of the δ-function.

In brief, if we take the complex conjugate of (5.2), the inverse transform

$$f^*(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} g^*(\omega) \exp(-i\omega' t)\, d\omega'$$

The intensity I of a function is defined to be

$$I = \int_{-\infty}^{\infty} |\, f(t)|^2 \, dt = \int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} dt * (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega\, g(\omega) \exp(i\omega t)$$

$$* (2\pi)^{-1/2} \int_{-\infty}^{\infty} d\omega'\, g^*(\omega') \exp(-i\omega' t)$$

$$= \int_{-\infty}^{\infty} d\omega\, g(\omega) \int_{-\infty}^{\infty} d\omega'\, g^*(\omega')\, \delta(\omega - \omega')$$

$$= \int_{-\infty}^{\infty} d\omega\, g(\omega)\, g^*(\omega) = \int_{-\infty}^{\infty} |\, g(\omega)|^2 \, d\omega$$

$|g(\omega)|^2$ is sometimes denoted as $\varphi_{tt}(\omega)$ or alternatively $S(n)$ and is called the power density spectrum or power spectral density.

Example: Damped harmonic oscillator

$$f(t) = 0 \quad , \quad t < 0$$

$$= \exp(-t/\sigma)\sin(\omega_n t), \quad t \geq 0$$

$$g(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$$

$$= 1/2\left[\frac{1}{(\omega + \omega_n - (i/\sigma))} - \frac{1}{(\omega - \omega_n - (i/\sigma))}\right]$$

$|g(\omega)|^2$ represents the energy (dissipated) per unit frequency. If $\sigma\omega_n >> 1$, then for $\omega \approx \omega_n$

$$|g(\omega)|^2 \approx \frac{k}{(\omega - \omega_n)^2 + (1/\sigma^2)}$$

which can be recognised as the response of a damped harmonic oscillator to driving frequencies near to its resonant frequency.

Note that the term I, defined in section 5.4 is the mean square value (or variance) of the function (often a time series). In other words the area under a power spectral density function *S(n)* is simply the variance.

It is often found convenient to plot *S(n)* against the logarithm of frequency. In order to preserve the equivalence of areas under the curve with contributions to the variance, the y-axis is chosen as *n S(n).*

5.5 Frequency response, convolution and deconvolution

For a linear system with transfer function *G(s)* we can examine the response of the system to a sinusoidal input $u(t) = \sin(\omega t)$ using the techniques developed in section 3.

$$U(s) = \frac{\omega}{s^2 + \omega^2}$$

The output $C(s) = G(s)\,U(s)$. Taking a ratio of suitable polynomial expansions for *G(s)* and using partial fractions to express *C(s)* gives

$$C(s) = \frac{G(s)\omega}{s^2 + \omega^2} = \frac{A_1}{s - i\omega} + \frac{A_2}{s + i\omega} + \frac{B_1}{s - p_1} \ldots + \frac{B_n}{s - p_n}$$

where, it can be recalled, the p_i are the poles of the system (roots of the characteristic equation), and A_i, B_i are simply constants.

Taking inverse Laplace transforms gives

$$C(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} + B_1 e^{p_1 t} \ldots + B_n e^{p_n t}$$

The first two terms represent the particular integral (often referred to as the steady state response) and the remaining terms represent the complementary function (the transient response).

Assuming all the poles p_i have negative real ports (true for stable systems) the transient terms tend to zero as t tends to infinity, thus only the first two terms remain, ie

$$c(t) \Big|_{t\to\infty} = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

A_1 and A_2 can be determined by multiplying $C(s)$ by $(s - i\omega)$ and $(s + i\omega)$ respectively and then setting $s = i\omega$ and $s = - i\omega$.

$$A_1 = \left[\frac{(s - i\omega)\, G(s)\omega}{s^2 + \omega^2}\right]_{s = i\omega} = \frac{G(iw)}{2i}$$

Similarly $A_2 = G(-i\omega)/-2i$

Written in magnitude and phase terms (ratio of steady state output to input amplitude, and phase shift between output and input sin functions).

$$c(t)\,|_{t\to\infty} = |G(i\omega)|\, sin\,(\omega t + phase\ of\ G(i\omega))$$

In other words the magnitude and phase components of a systems frequency response is obtained by replacing s with $i\omega$ in the transfer function and taking the modulus and argument of the result.

If a system is exposed to a signal with a known spectral density $X(\omega)$, the output spectral density can be shown to be given by

$$Y(\omega) = |G(i\omega)|^2\, X(\omega)$$

where $|\ G\ (i\omega)\ |^2$ is sometimes written as $H(\omega)$.

This is often written in alternative notation as

$$Y(n) = S(n)\, X(n)$$

If $f(x)$ and $g(x)$ are two functions, the convolution, written $f * g$ is defined to be

$$h(z) = \int_{-\infty}^{\infty} f(x)\, g(z - x)\, dx$$

Note that if g is the delta-function, then $h(z) = f(z)$.

The Fourier transform of the product of two functions is the convolution of the separate Fourier transforms multiplied by $(2\pi)^{-1/2}$.

$$g(\omega) = (2\pi)^{-1/2} \int f_1(t)\, f_2(t) \exp(-i\omega t)\, dt$$

$$= (2\pi)^{-1/2} \int dt\, f_2(t) \exp(-i\omega t)$$

$$* (2\pi)^{-1/2} \int d\omega'\, g_1(\omega') \exp(i\omega' t)$$

$$= (2\pi)^{-1} \int d\omega'\, g_1(\omega') \int dt\, f_2(t) \exp[-i(\omega - \omega')t]$$

$$= (2\pi)^{-1/2} \int d\omega'\, g_1(\omega')\, g_2(\omega - \omega')$$

$$= (2\pi)^{-1/2} g_1 * g_2$$