# 6. Matrices and Eigen Value Analysis

Notation: Bold capitals, **A**, **M** denote matrices.

Elements/components of matrix denoted by lower case with subscripts:

$$\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ . & & & . \\ . & & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is an *m* by *n* matrix, *(m x n)*

Lower case bold is used for column or row vectors (matrices with one of the dimensions equal to unity), eg **u, v, x y**.

For square matrices: The *determinant* of ***M*** *(det **M**)* is written $|\boldsymbol{M}|$

The *transpose* of ***A*** is $\boldsymbol{A}^T$, *ie* $a_{ij} \leftrightarrow a_{ji}$

***M*** is *singular* if $|\boldsymbol{M}| = 0$, otherwise *non-singular*

*Inverse* of ***A*** for non-singular matrices is denoted $\boldsymbol{A}^{-1}$ such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$, the *unit matrix* .

A matrix ***M*** is *symmetric* if $\boldsymbol{M}^T = \boldsymbol{M}$

If $\boldsymbol{M}^T = -\boldsymbol{M}$, then the matrix is *skew-symmetric*.

It can be shown that $(\boldsymbol{A}\,\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T$

If $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$ the matrix is *orthogonal*

For complex matrices $\boldsymbol{A}^*$ represents taking the complex conjugate of each element.

$(\boldsymbol{A}^T)^* = (\boldsymbol{A}^*)^T$ is denoted by $\boldsymbol{A}^+$ and called the *Hermitian conjugate* of ***A***.

A matrix is *Hermitian* if $\boldsymbol{A}^+ = \boldsymbol{A}$.

## 6.1 Eigenvalues and Eigenvectors

For a non-singular matrix $\boldsymbol{A}$ and two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of dimension $n$ with $\boldsymbol{A}$ being $n$ x $n$, we can write

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{y}$$

which represents a transformation of $x$ onto $y$. The inverse transformation is given by

$$\boldsymbol{y} = \boldsymbol{A}^{-1}\boldsymbol{x}$$

If $\boldsymbol{x}$ is simply a constant multiple of $y$, ie $x = \lambda \boldsymbol{y}$, then

$$\boldsymbol{A}\boldsymbol{y} = \boldsymbol{x} = \lambda\boldsymbol{y} \qquad (6.1)$$

A non-zero vector $\boldsymbol{y}$ which satisfies equation (6.1) for some value of $\lambda$ is called an **eigenvector** of $\boldsymbol{A}$, and $\lambda$ is the corresponding *eigenvalue.*

The eigenvector is the discrete analogue, in relation to a matrix, of the continuous eigenfunction of a differential operator.

Denote the ith eigenvector by $\boldsymbol{y}^i$ and the corresponding eigenvalue by $\lambda\lambda_i$ for i = $1, 2, \ldots n$.

$$\boldsymbol{A}\boldsymbol{y}^i = \lambda\lambda_i \boldsymbol{y}^i \qquad (6.2)$$

It turns out that an $n$ x $n$ matrix has $n$ eigenvectors and eigenvalues.

It can be shown that the eigenvalues of a Hermitian matrix are real and that eigenvalues corresponding to different eigenvalues are orthogonal.

To solve (6.1) we proceed as follows:

$$\boldsymbol{A}\boldsymbol{y} - \lambda\lambda\boldsymbol{y} = 0$$

$$\Rightarrow \quad \boldsymbol{A}\boldsymbol{y} - \lambda\lambda\boldsymbol{I}\boldsymbol{y} = (\boldsymbol{A} - \lambda\boldsymbol{I})\,\boldsymbol{y} = 0$$

This can be solved (with non-trivial result) only if

$$|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0 \qquad (6.3)$$

ie for non-zero eigenvectors. This is an application of the general result:

The equation $\boldsymbol{Ax} = 0$, can have a solution with $\boldsymbol{x} \neq 0$ only if $|\boldsymbol{A}| = 0$

Equation 6.3 is known as the *characteristic equation* for **A**. It is a polynomial equation in $\lambda$ of degree $n$. Its roots are the eigenvalues $\lambda_i$. The $\boldsymbol{y}^i$ corresponding to the $\lambda_I$ are the eigenvectors of **A**.

The *trace* or *spur* of a matrix **A** is defined to be $\sum_i a_{ii}$. It can be shown that $\sum_i \lambda_i$ is equal to the trace of the original matrix **A**.

It can be shown that eigenvectors of a Hermitian matrix can always be made to be mutually orthogonal (*normalised*).

Normalised eigenvectors can be used to diagonalise matrices. To diagonalise a matrix **A** we are looking for a matrix **U** such that $\boldsymbol{U}^T \boldsymbol{A} \boldsymbol{U}$ is a *diagonal* matrix $\boldsymbol{\Lambda}$ *(ie* $a_{ij} = 0 \ \forall i \neq j$*).*

It can be shown that simply taking the elements of **U** according to

$$u_{ij} = (\boldsymbol{y}^j)\ i = y_i^j \qquad (6.4)$$

is sufficient. The transformation $\boldsymbol{U}^T\boldsymbol{AU}$ is called an orthogonal transformation. It is a special case of the general similarity transformation of **A** given by $\boldsymbol{S}^{-1}\boldsymbol{AS}$ for a non-singular matrix **S**

We are now in a position to use the theory developed to examine dynamical systems.

**Example**. Normal frequencies and modes of oscillation of three particles of masses, *m, µm, m* connected in that order in a straight line to two equal light springs of force constant *k*.

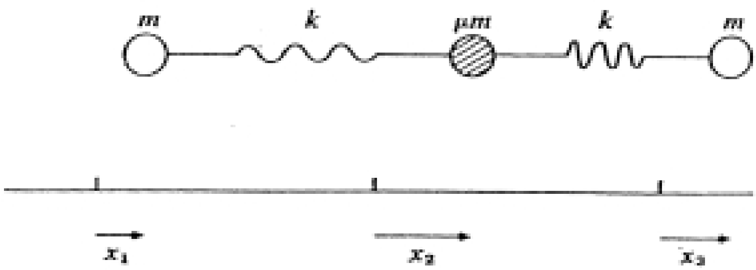


*Fig 6.1 The coordinate system for the example. The three masses, m, µm and m are connected by two equal light springs of force constant k.*

The situation is shown in Fig 6.1 in which the coordinates of the particles $x_1, x_2, x_3,$ are measured from their equilibrium positions at which the springs are neither extended nor compressed.

The kinetic energy of the system is simply

$$T = \tfrac{1}{2}m(\dot{x}_1^2 + \mu\dot{x}_2^2 + \dot{x}_3^2)$$

whilst the potential energy stored in the springs is

$$V = \tfrac{1}{2}k[(x_2 - x_1)^2 + (x_3 - x_2)^2]$$

The kinetic and potential energy symmetric matrices are thus

$$\mathbf{A} = \frac{m}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \frac{k}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

To find the normal frequencies we have, following 6.3, to solve $|\boldsymbol{B} - \omega^2 \boldsymbol{A}| = 0$. Thus writing $m\omega^2/k = \lambda$ we have

$$0 = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\mu\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

which leads to

$$\lambda = 0, 1, \text{ or } 1 + (2/\mu)$$

The corresponding eigenvectors are (respectively)

$$\mathbf{X}^1 = 3^{-1/2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{X}^2 = 2^{-1/2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{X}^3 = [2 + (4/\mu 2)]^{-1/2}\begin{bmatrix} 1 \\ -2/\mu \\ 1 \end{bmatrix}. \quad (6.5)$$

The physical motions associated with these solution are illustrated in fig 6.2. The first, λ $\lambda$= ωω= 0 and all the $x_i$ equal, merely describes the bodily translation of the whole system, with no (i.e. zero frequency) internal oscillations.

In the second solution the central particle remains stationary, $x^2{}_2 = 0$, whilst the other two oscillate with equal amplitudes in antiphase with each other. This motion of frequency $\omega = (k/m)^{1/2}$ is illustrated in the middle of fig 6.2.

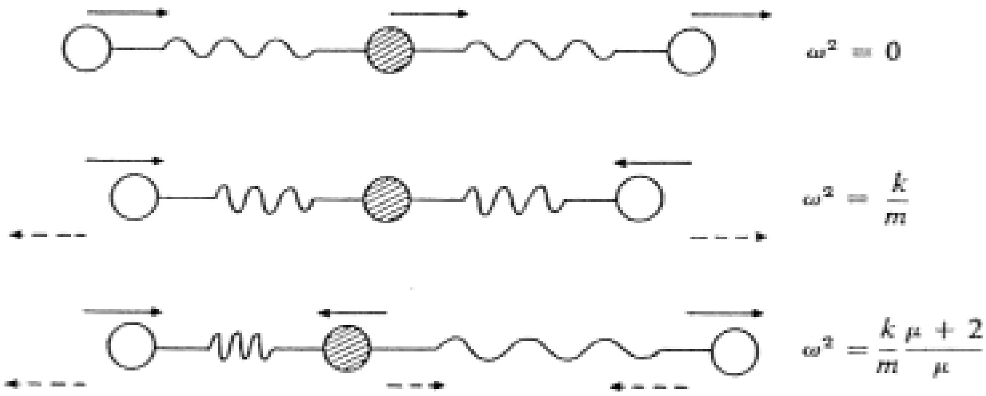


*Fig 6.2 The normal modes of the masses and springs*

The final and most complicated of the three normal modes has a frequency $w = (k(\mu + 2)/m\mu)^{1/2}$, and involves a motion of the central particles which is in antiphase with that of the two outer ones and has an amplitude which is $2/\mu$ times as great. In this motion the two springs are compressed and extended in turn.

The eigenvectors $\boldsymbol{x}^k$ obtained by solving $(\boldsymbol{B} - \omega^2 \boldsymbol{A})\boldsymbol{x} = 0$ are not mutually orthogonal unless $\boldsymbol{A}$ is a multiple of the unit martrix [$\mu = 1$], but it can be shown that they do satisfy

$\boldsymbol{x}^{Ti}\mathbf{A}\boldsymbol{x}^j = O$ and $\boldsymbol{x}^{Ti}\mathrm{B}\boldsymbol{x}^j = O$ for $i \neq j$