1. Velocity and Acceleration

[This material relates predominantly to modules ELP034, ELP035]

1.1 Linear Motion

Linear displacement and distance

The linear displacement is the length moved in a given direction - it is a **vector** quantity.

The magnitude of the displacement is the distance - a **scalar** quantity.

Linear velocity and speed

The linear velocity is the rate of change of displacement with time. As displacement is a **vector** so velocity is a **vector**.

The magnitude of the velocity is **speed**. It is the rate of change of distance with time - hence it is a **scalar**.

If a body moves with uniform velocity then it must move in a fixed direction with constant speed.

The average speed of a body is the total distance moved divide by the total time taken.

Distance - time curve

A graph plotted for distance (s) against time (t), might look like that in Figure 1.1:

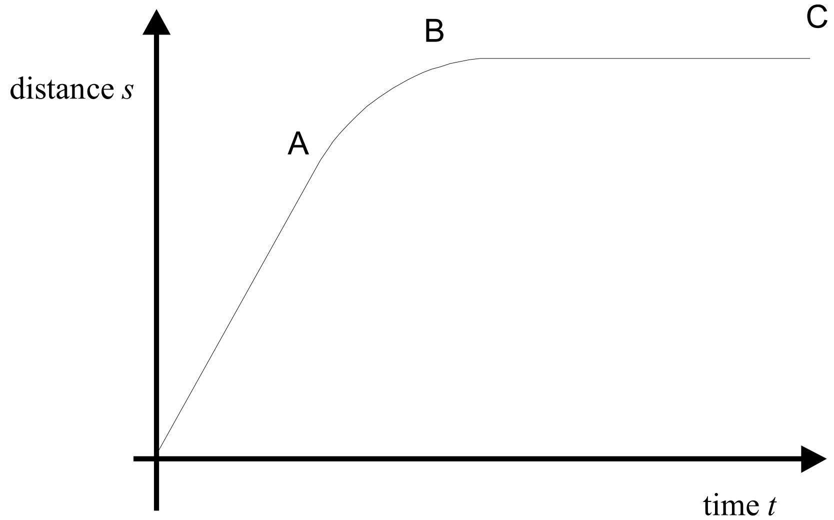

Figure 1.1: Distance-Time Curve

As speed is rate of change of distance with time, the slope, *gradient*, of the *s/t* curve is the speed.

Over the linear section OA of the curve the speed must be uniform.
Between A and B the gradient is becoming less and less, hence the body is slowing down.
At B the body is stopped (distance is not increasing) and remains at rest between B and C.

Linear acceleration

The linear acceleration of a body is the rate of change of linear velocity with time. It is a **vector**.
If acceleration is uniform the speed must be increasing by equal amount in equal time intervals.

Worked example 1.1

A car is travelling along a straight road at 13 m/s. It accelerates uniformly for 15 s until it is moving at 25 m/s

Solution

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time taken}} = \frac{dv}{dt}$$
$$= \frac{25-13}{15}$$
$$= 0.8 m/s^2$$

Speed - Time curve

A graph of speed (v) of a body plotted against time (t) might be as shown by the graph in figure 1.2:

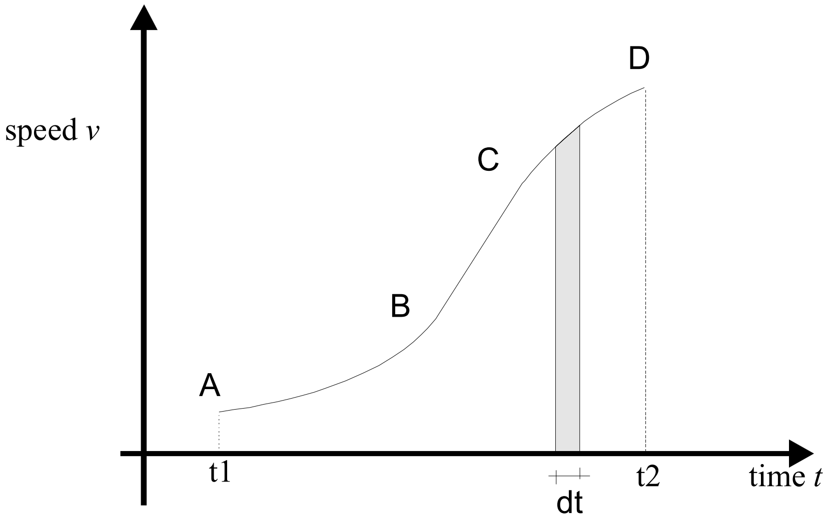

Figure 1.2: Speed-Time Curve

As acceleration is rate of change of speed (v) with time (t), the slope, *gradient*, of the v/t curve is the speed.

In Figure 1.2 the gradient between A and B is increasing - hence acceleration - is increasing, between B and C it is constant, between C and D it is decreasing.

If in the small time interval dt, the speed is v. The distance covered in the time dt is

$$ds = v\,dt$$

$$v = \frac{ds}{dt}$$

The total distance s travelled in the time interval between $t_1$and t_2 is the integral of this i.e.

$$s = \int ds = \int_{t_2}^{t_1} v\,dt$$

This integral is the same as the area under the curve.

Thus the distance travelled in any time interval is the area under the v/t curve between the start and end time.

Equations for linear uniformly accelerated motion

If a body that is moving in a straight line and started with initial speed u undergoes a *uniform* acceleration a for a time t until its velocity is v, then the speed time curve would look like that in Figure 1.3:

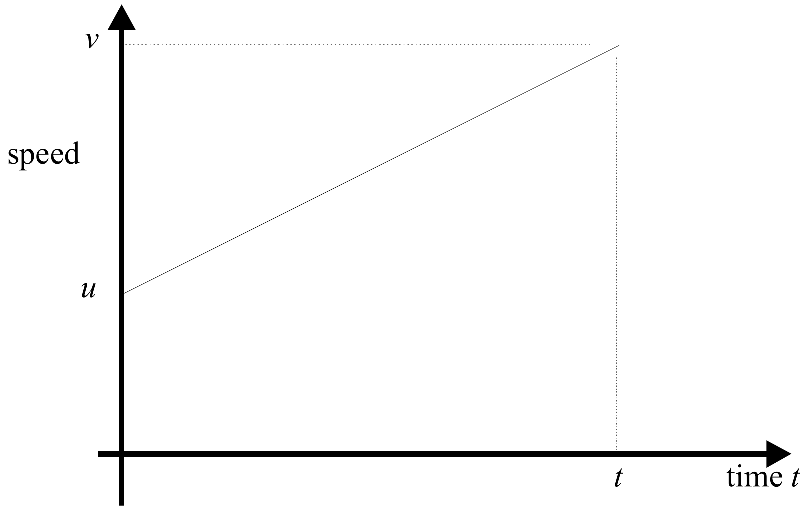

Figure 1.3: Uniformly accelerated linear motion

Since a is uniform, its magnitude is

$$a = \frac{\text{change in speed}}{\text{change in time}} = \frac{dv}{dt} = \frac{v-u}{t}$$

$$at = v - u$$

$$t = \frac{v-u}{a}$$

$$v = u + at$$

Equation 1.1

In this case, the *average speed* will be the speed at *t/2*. Hence:

$$\text{average speed} = \frac{u+v}{2}$$

Since the distance done = average speed × time then

$$s = \frac{(u+v)}{2}t$$

Equation 1.2

Substituting Equation 1.1 for v into Equation 1.2 gives

$$s = \frac{(u+u+at)}{2}t$$

$$s = u + \frac{1}{2}at^2$$

Equation 1.3

Substituting Equation 1.1 for t into Equation 1.2 gives

$$s = \frac{(u+v)}{2}\frac{(v-u)}{a}$$
$$2as = v^2 + u^2$$
$$v^2 = 2as - u^2$$

Equation 1.4

These four equations are the equations for **linear, uniformly accelerated motion**. They all contain 4 unknowns, you must know three before you can find the fourth.

Worked Example 1.2

A car starts from rest and accelerates in a straight line at 1.6 m/s^2 for 10s.

a) What is its final speed?
b) How far has it travelled in this time?

If the brakes are then applied and it travels a further 20m before stopping

c) What is the deceleration (retardation) ?

Solution

a)
Initial speed $u = 0$.
Acceleration $a = 1.6\ m/s$
Time $t = 10s$

Use Equation 1.1 to give v

$$v = u + at$$
$$v = 0 + 1.6 \times 10$$
$$v = 16\,m/s$$

b)
Use Equation 1.3 to give distance gone

$$s = ut + \frac{1}{2}at^2$$
$$s = 0 + \frac{1}{2}1.6 \times 100$$
$$s = 80\,m$$

c)

Initial speed $u = 16\ m/s$
Final speed $v = 0$
Distance $s = 20\ m$

Use Equation 1.4 to find acceleration

$$v^2 = u^2 + 2as$$
$$0 = 16^2 + 2 \times a \times 20$$
$$a = -6.4\, m/s^2$$

Deceleration is negative acceleration so

Deceleration = $6.4\ m/s^2$

Worked Example 1.3

A hoist starts at ground level and accelerates as 1.2 m/s^2 for 5s. It then moves with uniform speed for 10s and is finally brought to rest at the top of a building with a retardation (deceleration) of 2.0 m/s^2.

a) Draw the speed time graph of the motion
b) What is the height of the building?

Solution

a)
We can draw the speed - time curve shape as in Figure 1.4

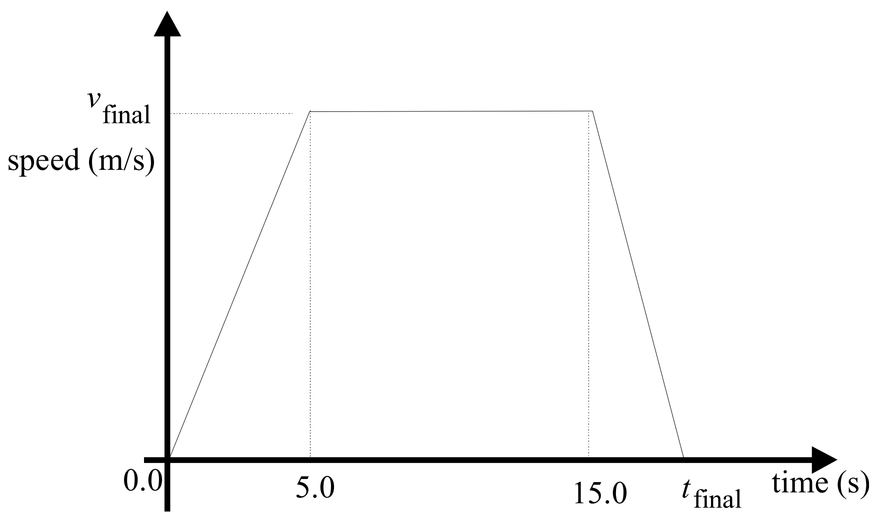

Figure 1.4: Speed-Time curve shape for Worked Example 1.3

But to complete this we need to find the unknowns, the final speed u and the time of the retardation.

Initial speed $u = 0.0$
Acceleration $a = 1.2\ m/s^2$
Time $t = 5\ s$

Use Equation 1.1 to give final speed

$$v = u + at$$
$$v = 0 + 1.2 \times 5$$
$$v = 6.0\, m/s$$

For the retardation time:
Initial speed $u = 6.0\ m/s$
Final speed $v = 0$
Acceleration $a = -2.0\ m/s^2$ (NOTE: Negative sign as retardation)

Using Equation 1.1 to give time

$$v = u + at$$
$$t = \frac{(v-u)}{a} = \frac{-6.0}{-2.0} = 3\,s$$

We can now complete the speed time curve by putting in the values, as shown in Figure 1.5

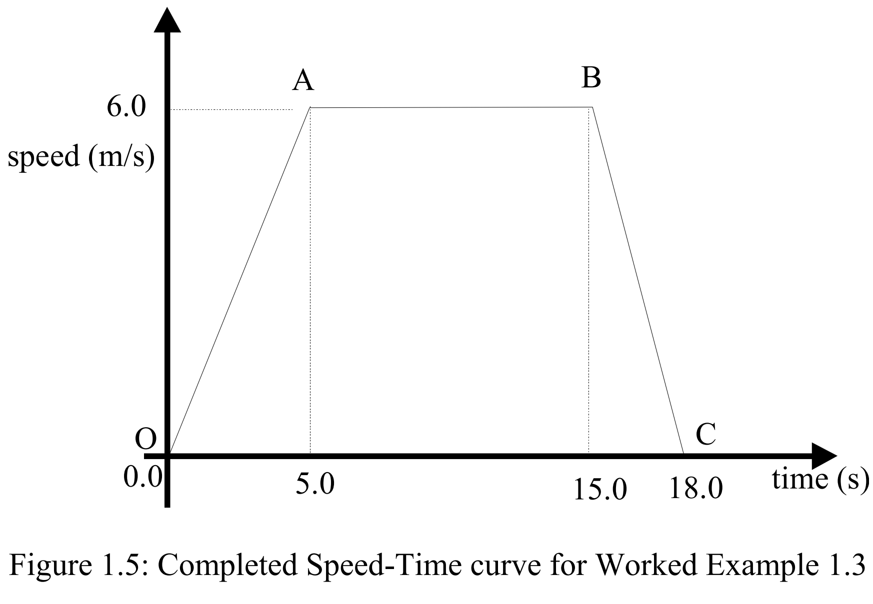

Figure 1.5: Completed Speed-Time curve for Worked Example 1.3

b)

The total distance travelled is the area under the speed-time curve. i.e. the area of the trapezium OABC

$$s = \left(\frac{5\times 6}{2}\right) + (10\times 6) + \left(\frac{3\times 6}{2}\right)$$
$$s = 84\,m$$

Thus the height of the building is 84m

Free fall under gravity

When a body falls to earth freely (without any other forces involved) the acceleration is called *acceleration due to gravity*. It is often given the symbol *g*.

Provided that air resistance is negligible all bodies, **heavy or light** fall at the **same** acceleration. Although *g* varies very slightly at different points on he earth $g = 9.81\ m/s^2$ can be used in most calculations, and will be used in the rest of these notes. (To four decimal places $g = 9.8142\ m/s^2$.)

Worked Example 1.4

A worker drops a hammer from the top of a 60m high building. If the speed of sound in air is 340 m/s, how long does the worker have to shout down to warn colleagues (if his warning is to reach them before the hammer!) Neglect air resistance.

Solution:

The solution is, the length of time for the hammer to reach the ground minus the length of time it takes for the shouted warning to reach the workers on the ground.

For the hammer:
Initial speed $u = 0$
Acceleration $a = 9.81\ m/s^2$
Distance $s = 60m$

Use Equation 1.3 to give time.

$$s = ut + \frac{1}{2}at^2$$
$$60 = 0 + \frac{1}{2}9.81t^2$$
$$t = \sqrt{12.2324} = 3.5\,s$$

For the shout

$$\frac{s}{t} = 340$$
$$t = \frac{60}{340} = 0.18\,s$$

Difference in travel time between hammer and shout:

$$3.5 - 0.18 = 3.32 \text{ s}$$

The warning must be shouted within 3.32 seconds of dropping the hammer.

1.2 Angular Motion

Angular speed and velocity

Consider a point P moving along a line QP as shown in Figure 1.6

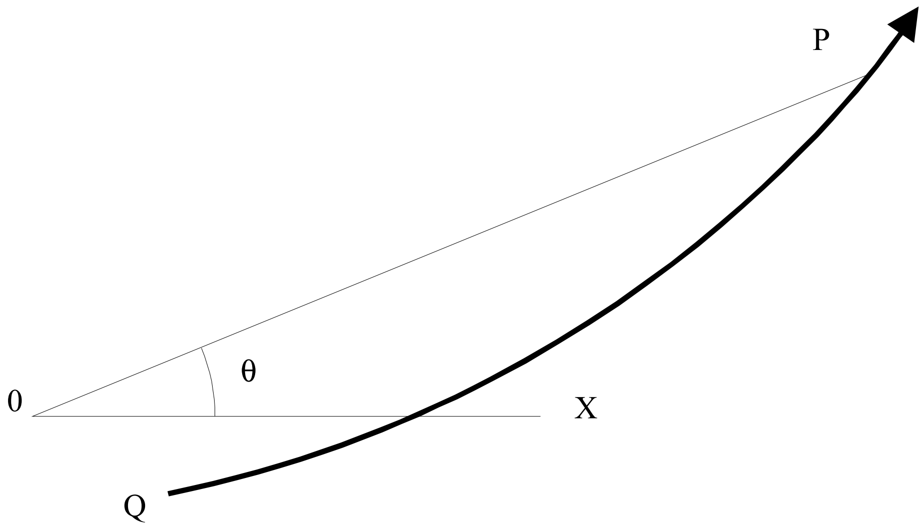

Figure 1.6: Angular motion

Let OX be a fixed line and θ be the angle made with OP at some time *t*. The angular velocity ω of P about O is the rate of change of θ with time in the sense of increasing θ.

Hence

$$\omega = \frac{d\theta}{dt}$$

Equation 1.5

When the direction of increase of θ is not included, $\frac{d\theta}{dt}$ is called angular speed.
The SI unit for angular speed is radians per second, *rad/s*.

Angular acceleration

If the angular velocity of the point P in Figure 1.6 is changing with time, then the angular acceleration, α, of P is the rate of change of its angular velocity:

$$\alpha = \frac{d\omega}{dt}$$

Equation 1.6

In the sense of increasing θ.

In SI units angular acceleration is measured in *rad/s²*.

If the angular acceleration is uniform, then when angular speed changes from ω_1 to ω_2 in time t, its magnitude is

$$\alpha = \frac{\omega_2 - \omega_1}{t}$$

Equation 1.7

1.3 Relationship between linear and angular motion

Relationship between linear speed and angular speed

If a point P move round a circle of radius r with constant linear speed, v, (see Figure 1.7) then the angular speed, ω, will be constant at

$$\omega = \frac{\theta}{t}$$

Equation 1.8

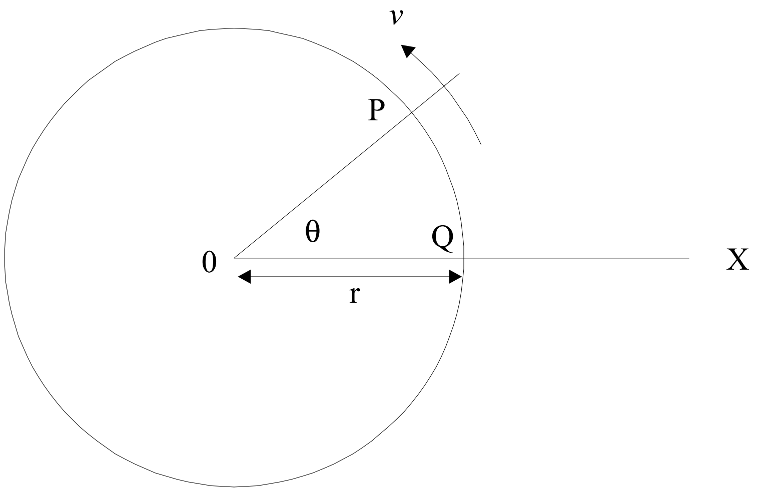

Figure 1.7: Circular Motion

Where t is the time to move from Q to P along the arc QP of the curve.

However, arc length QP is $r\theta$ when θ is measured in radians. Hence linear speed v is

$$v = \frac{\text{length of arc QP}}{t} = \frac{r\theta}{t}$$

Equation 1.9

Substituting Equation 1.8 into Equation 1.9 leads to the this relationship for circular motion:

$$v = r\omega$$
$$\text{linear speed} = \text{radius} \times \text{angular speed}$$

Equation 1.10

Relationship between angular speed and frequency of rotation

Let P in Figure 1.7 rotate with constant frequency of *n rev/s*.
Since for each revolution turned the angle is 2π rad, then the number of radians turned through per second is $2\pi n$. This is the angular speed. So we can write

$$\omega = 2\pi n$$
$$\text{angular speed} = 2\pi \times \text{revolutions per second}$$

Equation 1.11

Relationship between linear acceleration and angular acceleration

By Equation 1.6 and Equation 1.10

$$\alpha = \frac{d\omega}{dt}$$

and

$$v = r\omega$$

$$\alpha = \frac{d}{dt}\left(\frac{v}{r}\right)$$

as r is constant this can be written

$$\alpha = \frac{1}{r}\frac{dv}{dt}$$

and as $\frac{dv}{dt}$ is **linear acceleration** a,

$$\alpha = \frac{a}{r}$$
$$a = r\alpha$$

Equation 1.12

1.4 Uniform circular motion - (acceleration)

Consider again a point X moving round a circle, radius r, at uniform speed v.
Since the direction of motion is changing form instant to instant, the velocity is changing and hence point X has an acceleration.

Let AB represent the velocity vector when X is at point Q and AC represent the velocity vector when X is at point P, as shown in Figure 1.8. The velocity vectors at points Q and P are tangential to the circle of motion, sides AB and AC are parallel to these vectors.

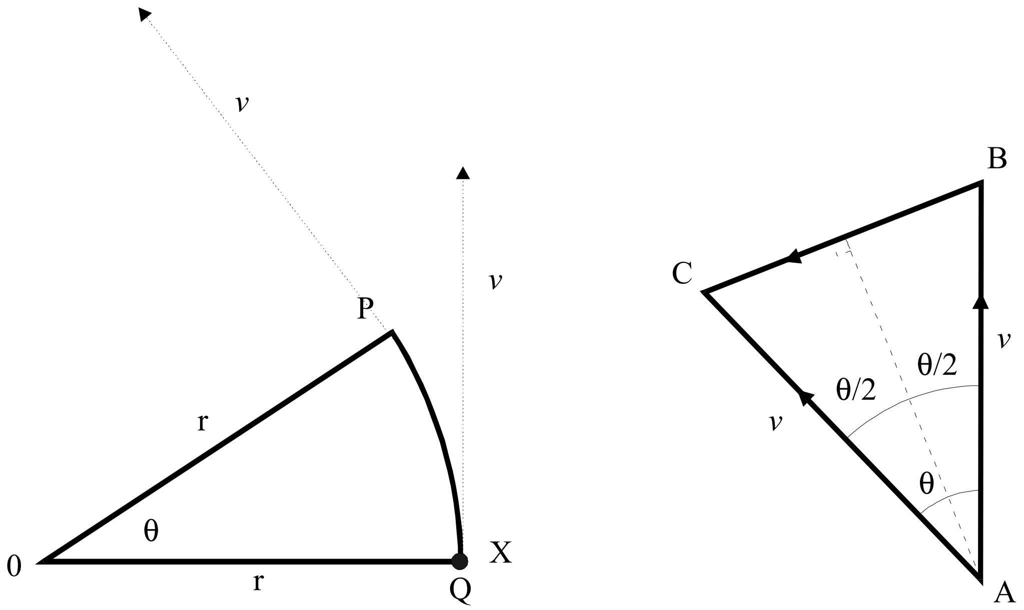

Figure 1.8: Angular Acceleration

The line BC represents the change in velocity as the point moves from Q to P.
The average acceleration of P is

$$a = \frac{\text{length of BC}}{\text{time to travel from Q to P}}$$

$$\text{length of arc QP} = r\theta$$

So

$$\text{time to travel from Q to P} = r\theta / v$$

Also

$$\text{length of BC} = 2v\sin(\theta/2)$$

$$a = \frac{v^2}{r}\frac{\sin(\theta/2)}{\theta/2}$$

To find the acceleration of P at Q, let the angle $\theta/2$ tend to zero, so

$$\sin(\theta/2) \to \theta/2$$

and the acceleration is given by

$$a = \frac{v^2}{r}$$

Equation 1.13

When $\theta/2 \to 0$, the direction of BC approaches the direction of QO, that is, towards the centre of the circle.
Hence, the acceleration of a point moving round a circle with radius *r*, at constant speed *v* is v^2/r towards the centre of the circle.

Worked Example 1.5

A flywheel, diameter 1.1m, rotating at 1200 rev/min slows down at a constant rate to 900 rev/min in 30 s.

Find:
a) The initial angular speed
b) The final angular speed
c) The angular acceleration
d) The initial speed of a point on the rim of the flywheel.

Solution

a)
From Equation 1.11 angular speed $\omega = 2\pi n$

Initial n = *1200 rev/min = 1200 /60 = 20 rev / s*

$$\omega = 2\pi \times 20 = 125.7\, rad/s$$

b)
Final n = *900 rev/min = 900/60 = 15 rev/s*

$$\omega = 2\pi \times 15 = 94.2\, rad/s$$

c)
Use Equation 1.7

$$\alpha = \frac{\omega_2 - \omega_1}{t} = \frac{94.2 - 125.7}{30}$$
$$\alpha = -1.05\, rad/s$$

d)
Equation 1.10 gives the linear speed, so initial speed on edge of flywheel is

$$v = r\omega$$
$$v = \frac{1.1}{2}125.7 = 69.1\, m/s$$

Worked Example 1.6

The spin dryer in a washing machine is a cylinder with diameter 500mm. It spins at 900 rev/min.

Find
a) the speed of a point on the edge of the drum
b) the acceleration of a point on the edge of the drum

Solution

a)
Frequency of rotation in rev/s

$$n = 900/60 = 15\ rev/s$$

From Equation 1.11

$$\begin{aligned}\omega &= 2\pi n \\ &= 2\pi \times 15 \\ &= 94.2\, rev/s\end{aligned}$$

From Equation 1.10 the linear speed is

$$v = r\omega$$

$$v = \frac{500 \times 0.001}{2} 94.2 = 23.6\, m/s$$

b)
From Equation 1.13

$$a = \frac{v^2}{r} = \frac{23.6^2}{(500 \times 0.001)/2} = 2230\, m/s$$

Note how large this is, it is 227 times the acceleration due to gravity!